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ANALYTICAL GEOMETRY
AND
CONIC SECTIONS
BY
HANN & YOUNG

A

RUDIMENTARY TREATISE

ON

ANALYTICAL GEOMETRY

AND

CONIC SECTIONS.

BY

JAMES HANN,

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Honorary Member of the Philosophical Society of Newcastle-upon-Tyne.

NEW EDITION, REWRITTEN AND ENLARGED

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PREFACE.

STUDENTS of this Rudimentary Treatise are expected to bring to the perusal of it a previous acquaintance with the principles of Algebra, and of Plane Trigonometry. The former branch of science is more especially requisite; since the subject, as its name implies, is the application of algebra to geometrical inquiries. The learner cannot proceed far in the study of Analytical Geometry without feeling that enlarged views of the capabilities and importance of common algebra are gradually opening to him; that it is *here* presenting itself to him under an entirely new aspect; and that what he may have hitherto regarded merely as an ingenious and useful contrivance, for abridging, expediting, and generalising, numerical calculations, proves itself, in this application of its principles, to be a valuable instrument of investigation in fields of research which had seemed to lie altogether outside its own legitimate province.

This extension of algebra beyond the limits prescribed to it in books expressly on the subject, cannot fail to be a matter of interest to the earnest and inquiring student; and in observing the way in which certain remarkable theorems of geometry are educed from algebraical expressions, he must experience a sort of fascination, which, on the score of intellectual gratification alone, will amply repay him for the time and thought expended upon ANALYTICAL GEOMETRY. Under this term, in its full extent, much more, however, is comprehended than is given in the present Rudimentary Treatise, which is restricted to the geometry of the CONIC SECTIONS. The curves so called, and the properties of them here established, are in very frequent request in various departments of natural philosophy; and in physical astronomy they are all-important.

Although this volume is published as a new edition of Mr. Hann's book, yet but little of the original work has been retained. However well designed, the book was somewhat negligently put together; and therefore, while preserving the general plan, I have thought it better, in the interest of the learner, to re-write the treatise. It is necessary, as well in justice to Mr. Hann as to myself, that this should be explicitly stated; so that it may be distinctly understood that for everything in the book I alone am responsible.

J. R. YOUNG.

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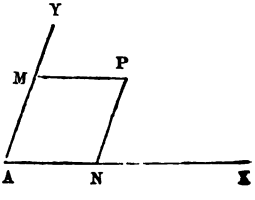
ANALYTICAL GEOMETRY.

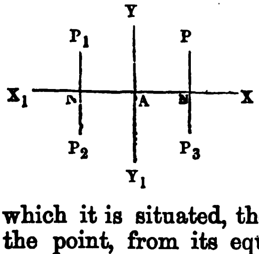
Analytical Geometry, or Algebraical Geometry, as it is sometimes called, is the application of algebra to the investigation of the properties of geometrical lines and surfaces. It shows that all such lines and surfaces may be as significantly represented by algebraical equations as by geometrical figures; and that the various properties of them may be deduced from these equations by the ordinary processes of common algebra. As at the outset of the investigation the geometrical conditions are translated into algebra, so the successive algebraical deductions may be translated back into geometrical theorems—the necessary consequences of those conditions. The subject is as interesting as it is useful; and there is perhaps no department of mathematics which, to the zealous student, possesses such attractiveness as Analytical Geometry.

Section I.—EQUATIONS OF A POINT.



The position of a point P, in the plane of the paper, is said to be *geometrically given* when the place of it is actually marked by a visible dot; its position is said to be *analytically* or *algebraically given*, when, without this visual aid, the linear distance of it from each of two intersecting straight lines, AX, AY, is made known. If x represents, in general, the linear distance PM of any point P from the fixed line AY, and y its linear distance PN from the other fixed line AX, these linear distances being measured from P along parallels to AX, AY, respectively, it is clear that the point will become geometrically determinable; we shall only have to measure the given length (x) along AX, from A, and the given length (y) along AY, and then from the terminations (N and M) of these lengths, to draw a straight line parallel to AY, and a straight line parallel to AX; for the intersection of these lines must be the point, and the only point, within the angular space YAX, to which the stated measures apply. If, in the present instance, AN be *three* of the assigned units of length, and AM *four*, then $x = 3$, $y = 4$, would be what are called *the equations of the point* P. The length x is called the *abscissa* of the point, and the length y the *ordinate* of it; when spoken of together, x, y, that is in this case, 3, 4, are called the *co-ordinates* of the point; AX, AY, are *the axes of co-ordinates*; the former, the axis of abscissas; and the latter, the axis of ordinates; or more briefly, the former the axis of x, and the latter the axis of y; and the point A, where the axes meet, or at which they intersect, and from which the co-ordinates are measured, is called *the origin of co-ordinates.*

If the axes be prolonged through the origin, there will be *four* angular compartments about the origin A, as in the annexed diagram, in each of which a point may be situated so that the absolute lengths of the co-ordinates of each may be the same—the points P, P_1, P_2, P_3, for instance, may each of them have the same co-ordinates, in so far as the mere *lengths* of the lines so called are concerned, and therefore, unless there be some means of denoting the compartment in which it is situated, the determination of the position of the point, from its equations, would be ambiguous—we should not know *which* of the *four* points, each having the same co-ordinates, is *the* point. To obviate this ambiguity, every abscissa measured from A towards X, and every ordinate measured from A towards Y, is regarded as *positive*, [illegible]ue taking the sign +; while abscissas measured [illegible]wards X_1, and ordinates from A towards Y_1, being



pposite in direction to the former, take, of course, the pposite sign, and are regarded as *negative*. Hence, calling he co-ordinates, as to mere *lengths*, a and b, we have—

For the equations of the point P, $x = a$, $y = b$;
" " " " P_1, $x = -a$, $y = b$;
" " " " P_2, $x = -a$, $y = -b$;
" " " " P_3, $x = a$, $y = -b$;

he *plus* sign being, of course, understood where no sign at ll is prefixed. And in this way the point is algebraically epresented, without any ambiguity, in whichever of the our compartments it may be situated. It *may* be situated n one of the axes, on the axis of x, for instance; or it may e situated on the axis of y; in the former case, its equaions are $x = a$, $y = 0$; or $x = -a$, $y = 0$, according as it s to the right or the left (in the preceding diagram) of the rigin; if the point be upon the axis of y, then its equations re $x = 0$, $y = b$, or $x = 0$, $y = -b$, according as it is bove or below the origin, in the foregoing diagram; but if he point be at the same time upon *both* axes, that is, at he origin A, then its equations are $x = 0$, $y = 0$.

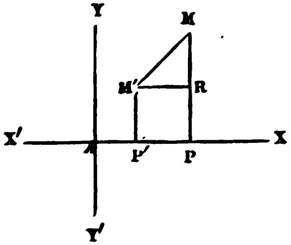
In analytical investigations it is usual to refer to a fixed solated point, as the point (a, b); and when the point o be distinguished from the other points is in a *line*, it is lenoted as the point (x', y'), or (x'', y''), &c., the notation $x, y)$ being used to denote *any* point in the line indiffeently. As points are thus always represented by their o-ordinates, in Analytical Geometry, the science is freuently called *Co-ordinate Geometry*.

PROBLEM.—*To find the analytical expression for the distance between two given points.*

By the points being *given* is of course meant that their o-ordinates are given. These may be measurements in eference to rectangular axes, or in reference to oblique xes.

FIRST: *When the axes of reference are rectangular.* Let he points M and M′ be represented by (x', y'), and (x'', y''), he axes of reference being rectangular; or, which is the ame thing, let (x', y'), (x'', y''), be the rectangular cordinates of the two points; and let the distance between hem (MM′) be denoted by D.

Draw MP, M′P′, parallel to the axis of y, and M′R parallel to the axis of x.



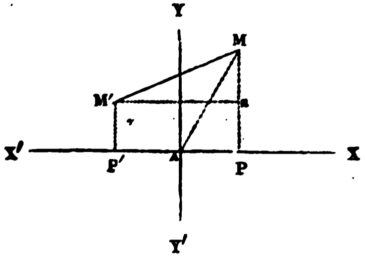
Then (*Euclid* i., 47)

$$\begin{aligned} \text{MM}'^2 &= \text{M}'\text{R}^2+\text{RM}^2 = \text{P}'\text{P}^2+\text{RM}^2 \\ &= (\text{AP}-\text{AP}')^2+(\text{PM}-\text{PR})^2 \\ &= (x'-x'')^2+(y'-y'')^2 \\ \therefore\ \text{D} &= \surd\{(x'-x'')^2+(y'-y'')^2\} \quad (1) \end{aligned}$$

which is the general expression for the distance D, of two points (x', y'), (x'', y''), wherever they may be situated in the plane of the rectangular axes. If only one of the points M, or (x', y'), be situated in the compartment YAX, and the other M′, or (x'', y''), be in the compartment YAX′, then x'' will be negative; and the expression for their distance will be

$$\text{D} = \surd\{(x'+x'')^2+(y'-y'')^2\} \quad . \quad . \quad . \quad (2),$$

for the diagram will then be this—



in which

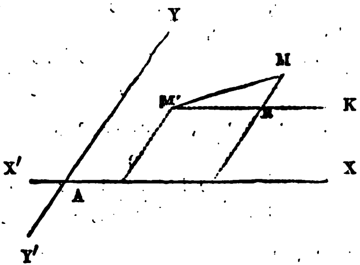
$$\begin{aligned} \text{MM}'^2 &= \text{M}'\text{R}^2+\text{RM}^2 = \text{P}'\text{P}^2+\text{RM}^2 \\ &= (\text{AP}+\text{AP}')^2+(\text{PM}-\text{PR})^2 \\ &= (x'+x'')^2+(y'-y'')^2 \\ \text{and } \therefore\ \text{D} &= \surd\ \{x'+x'')^2+(y'-y'')^2\}. \end{aligned}$$

And, in like manner, let the points be situated wherever they may in the plane of the axes, the general formula (1) will always correctly express their distance apart, by introducing into it the given co-ordinates with their proper signs.

If one of the points, as M′, be at the origin A, then $x'' = 0$, and $y'' = 0$; so that, in this case, the expression for the distance is simply

$$D = \sqrt{(x'^2 + y'^2)}.$$

SECOND: *When the axes of reference are oblique.* Let AX, AY, be the oblique axes; then (*Trigonometry*, p. 52),



$$D^2 = MM'^2 = M'R^2 + RM^2 - 2M'R \,.\, RM \cos MRM'.$$

But $M'R = x' - x''$, and $RM = y' - y''$; and if the angle A of the oblique axes, or which is the same thing, if the angle MRK, be represented by β, we shall have

$$\cos MRM' = -\cos MRK = -\cos\beta;$$

$$\therefore D = \sqrt{\{(x'-x'')^2 + (y'-y'')^2 + 2(x'-x'')(y'-y'')\cos\beta\}} \quad (3)$$

which becomes identical to the expression (1) when $\beta = 90°$, because $\cos 90° = 0$.

We shall now give a few examples of the application of the foregoing general formulas to particular cases, chiefly for the purpose of familiarising the learner with those analytical expressions which in what follows will be in very frequent request. It may, however, be as well to apprise him here that Analytical Geometry, like the Geometry of Euclid, is occupied—not with individual and special cases —but with the investigation of *general* properties free from all numerical restrictions; and therefore comprehending every special case.

EXAMPLES.*

1. What is the distance between the two points (4, 5) and (2, 3); the co-ordinates being rectangular?

Referring to the general expression (1), at p. 4, or rather, writing down that expression from memory, we find upon giving to x', x'', the special values 4, 2; and to y', y'', the special values 5, 3, that the distance D is
$D = \sqrt{\{(4-2)^2+(5-3)^2\}} = \sqrt{(2^2+2^2)} = \sqrt{8} = 2\sqrt{2}$, the linear distance being equal to the hypotenuse of a right-angled triangle, of which the sides are each *two* of the linear units of measure, whatever these may be.

2. What is the distance between the two points (−4, 2) and (4, −5), the co-ordinates being rectangular?
Ans. $\sqrt{113}$

3. Two points referred to rectangular axes are (7, −2) and (7, 3): required, the distance between them? *Ans.* 5.

4. Required, the distance between the two points (0, 0) and (−3, −6), the co-ordinates being rectangular?
Ans. $3\sqrt{5}$.

5. Required, the distance between the two points (0, 2) and (−4, 0), the axes being rectangular? *Ans.* $2\sqrt{5}$.

6. Required, the distance between two points (1, 2), (2, 1), the axes of reference being oblique, and inclined to each other at an angle of 45°? [$\cos 45° = \frac{1}{2}\sqrt{2}$. *Trigonometry*, p. 15.] *Ans.* $\sqrt{(2-\sqrt{2})}$.

7. What is the distance of the point (−3, −5) from the origin, the axes being inclined at an angle of 30°? [$\cos 30° = \frac{1}{2}\sqrt{3}$. *Trigonometry*, p. 14.]
Ans. $\sqrt{(34+15\sqrt{3})}$.

8. Required, the distance between the two points (4, 0), (−3, 2), the oblique axes making an angle of 60°? [$\cos 60° = \frac{1}{2}$.] *Ans.* $\sqrt{39}$.

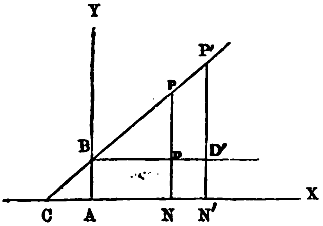
9. Required the distance between the two points (−3, −5), (4, −2), the co-ordinate angle being 120°?
Ans. $\sqrt{37}$

---

* The student is recommended to roughly sketch the position of the line, in reference to the axes, in each of these examples.

SECTION II.—EQUATION OF A STRAIGHT LINE.

FIRST: *When the axes of reference are rectangular.*—Let AX, AY, be the rectangular axes, and CP′ any straight line; then the parallels to the axes being drawn, as in the diagram, P being any point in the given straight line, we have



AN $= x$, PN $= y$. Let AB $= h$, then $\frac{\text{PD}}{\text{BD}}$ or $\frac{\text{P}'\text{D}'}{\text{BD}'} = \tan$ PBD (*Trigonometry*, p. 8), that is, $\frac{y-h}{x} = \tan$ PBD $= \tan$ C, the angle of inclination of the line CP′ to the axis of x. Call this tangent m; then

$$\frac{y-h}{x} = m, \therefore y-h = mx, \therefore y = mx+h \quad . \quad . \quad . \quad (1),$$

which is the general equation for a straight line. The constant co-efficient m is called *the co-efficient of inclination;* the other constant h is the ordinate AB of the point where the line cuts the axis of y. They are called *constants* because their values are fixed and invariable for the same straight line; and as the co-ordinates x, y, vary in value in passing from point to point of the line, these are called *variables*.

If the straight line passes through the origin, $h = 0$; and the equation is simply $y = mx$.

When the values of the constants m and h are given, we can always draw the particular line thus analytically represented; the positions of the axes being assigned, we have only from AY to cut off a length AB, numerically expressed by h, and then through B to draw CP inclined to CX at the angle whose tangent is the given number m, which angle is found against the number m in every Table of Natural Tangents. But a readier way of geometrically determining the straight line of which the algebraical equation is given will be shown presently.

The line AB is called the *intercept* of the axis of y; AC is the *intercept* of the axis of x; *this* intercept, found by putting $y=0$ in the equation of the line, is $x = \text{AC} = -\frac{h}{m}$.

If we divide each member of the equation (1) by h, it will become

$$\frac{y}{h} = \frac{m}{h}x + 1\text{; or, since } h = \text{AB}, \text{ and } m \text{ (or } \tan \text{C}) = \frac{\text{AB}}{\text{AC}},$$

the equation is

$$\frac{y}{\text{AB}} = \frac{1}{\text{AC}}x + 1;$$

or finally, putting a for the intercept AC, and b for the intercept AB, the equation is

$$\frac{y}{b} = \frac{x}{a} + 1, \text{ or } \frac{y}{b} - \frac{x}{a} = 1.$$

And this is the equation of the line CP in terms of the intercepts. The ordinary equation, $y = mx + h$, involves but a single intercept, that on the axis y, namely, $\text{AB} = h$. It is sometimes convenient that the intercept on the axis of x should enter instead: the equation would then be $y = m(x + \text{AC})$, for here when $y = 0$, $x = -\text{AC}$.

It will be instructive to the learner to consider the line in each of the four positions in the subjoined diagram; he will find that

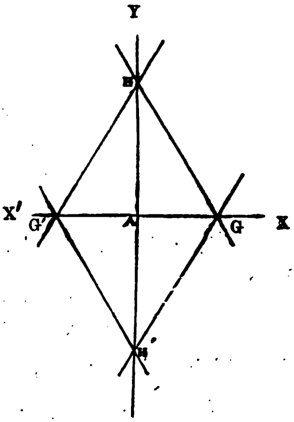
$y = -mx + h$, or $\frac{y}{b} + \frac{x}{a} = 1$ is the equation of HG

$y = +mx + h$, or $\frac{y}{b} - \frac{x}{a} = 1$ „ „ „ HG′

$y = +mx - h$, or $\frac{x}{a} - \frac{y}{b} = 1$ „ „ „ H′G

$y = -mx - h$, or $\frac{y}{b} + \frac{x}{a} = -1$ „ „ „ H′G′

In these equations, the numerical *value* only of the tangent of the inclination of the line to the axis of x is represented by m, the proper *sign* of this tangent being in each case prefixed, it being observed that the angle of inclination is always that which is situated *above* X′X, and has its opening to the *right hand*; in like manner the *lengths* only of h, and of the intercepts a, b, are represented by these symbols; the proper signs being in each case prefixed. If the symbols be put to represent the values, signs and all, the several distinct forms above may all be comprehended in one general equation, namely, in the equation



$$\frac{y}{b}+\frac{x}{a}=1 \quad . \quad . \quad . \quad (2),$$

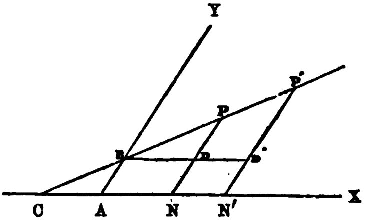
which, when a, b, are both positive, represents the line HG; when b is positive and a negative, the line HG′; when a is positive and b negative, the line H′G; and when both intercepts are negative, the line H′G′. And similar remarks of course apply to the other four distinct forms above; the general form (1), p. 7, comprehends them all.

If the straight line pass through the *origin*, then there are *no* intercepts, and the form (2) becomes inapplicable; but, as already remarked, the form (1), for this case, becomes $y = mx$; and the line may be drawn by putting *any* value x' for x, and thus determining y'; the line from the origin, through the point (x', y'), will be the line which the equation represents.

SECOND: *When the axes of reference are oblique.*—Let AX, AY, be the oblique axes, and CP′ any straight line; then the parallels to the axes being drawn, as in the diagram (p. 10), P being any point in the line, we have (*Trigonometry*, p. 52)

$$\frac{\mathrm{DP}}{\mathrm{BD}}=\frac{\sin \mathrm{PBD}}{\sin \mathrm{BPD}}=\frac{\sin \mathrm{PCX}}{\sin \mathrm{PBY}}, \text{ a constant ratio.}$$

If PCX, the inclination of the straight line to the axis of x, be represented by α, and YAX, the inclination of the axes



themselves, by β, the above ratio will be thus expressed, viz.,

$$\frac{\text{DP}}{\text{BD}}\left(\text{or }\frac{\text{AB}}{\text{AC}}\right)=\frac{\sin\alpha}{\sin(\beta-\alpha)};$$

for which m being put, and h for AB, the equation of the line is $y = mx+h$, the same as before; the only difference being that, in the case of rectangular axes, m is a *tangent*—the tangent of C; whereas in the case of oblique axes, it is the ratio of two *sines*, sin α to sin $(\beta-\alpha)$; that is, of sin PCX to sin PBY. And, therefore, the equation, in terms of the *intercepts*, is the same, viz.

$$\frac{y}{b}+\frac{x}{a}=1,$$

a being the intercept of the axis of x, and b the intercept of the axis of y. As in the case of rectangular axes, when the line passes through the origin, i.e. when $h=0$, the equation is $y=mx$.

EXAMPLES.

1. It is required to construct the line of which the equation in reference to assigned axes, whether rectangular or oblique, is $4y+x=6$?

Putting 0 for y, we have 6 for the corresponding value of x; and putting 0 for x, we have $1\frac{1}{2}$ for the corresponding value of y. Hence the line cuts the axis of x in the point $(6, 0)$, and the axis of y in the point $(0, 1\frac{1}{2})$: therefore, by measuring 6 of the given units of measure, from the origin, along the axis of x, and $1\frac{1}{2}$ of those units, from the origin, along the axis of y, we shall have two points on the axes, through which, if a straight line be drawn, it will be that represented by the given equation.

⁂ In the following examples the learner should actually sketch the positions of the lines.

2. Construct the equation $\text{A}x+\text{B}y+\text{C}=0$.

For $y=0$, we have $x=-\frac{\text{C}}{\text{A}}$; and for $x=0$, $y=-\frac{\text{C}}{\text{B}}$. These are the lengths and directions of the two *intercepts*, through the extremities of which the straight line represented by the equation passes.

We thus see that not only may every straight line, drawn in reference to assigned or assumed co-ordinate axes, be represented by a simple algebraical equation with two unknown quantities (x and y), but conversely, that every simple equation, with two unknown quantities, may always be represented by a straight line referred to assigned or assumed co-ordinate axes.

3. Determine the straight line of which the equation is $y=x-3$, and find its inclination to the axis of x, when the axes are rectangular.

Ans. The intercepts are $x'=3$, and $y'=3$.
The inclination is 45°.

4. Construct the straight line of which the equation is $y=5x$. *Ans.* The line passes through the origin, and through the point (1, 5).

5. Construct the straight line of which the equation is $3x-2y+4=0$.

Ans. The intercepts are $x'=-\frac{4}{3}$, and $y'=2$.

SECTION III.—THE STRAIGHT LINE SUBJECT TO CONDITIONS.

We have seen that the *general* equation of a straight line contains *two arbitrary constants*, that is, constants both of which become fixed in value only when some one, out of an indefinite number of straight lines, is singled out for algebraical representation. If but one of these disposable constants be fixed in value, the other remaining arbitrary, or open to any interpretation, as to value, which we may choose to give it, then the equation—though not restricted to a single line, to the exclusion of all other lines—will yet be limited to a distinct group of lines, to the exclusion of all other lines; so that, give whatever value we may to the other and still disposable constant, the particular line represented will always be a line belonging to this group.

One of the two constants may be so fixed in value that the lines of the group here alluded to shall all pass through one and the same fixed point; or both constants may be so determined that the single line represented shall pass through *two* assigned points. The present section will be devoted to the solutions of problems of this kind, that is, to the determining of the equations of straight lines subject to prescribed conditions.

PROBLEM 1.—*To find the equation of a straight line passing through a given point.*

Let the given point be denoted by (x', y'). Then since $y = mx+h$ denotes any straight line indifferently, so long as m and h are unrestricted in value, if one of the lines is to be subject to the condition of passing through the point (x', y'), the equation must be satisfied for these particular values of x and y; that is, the constants m and h must be so related that

$$y' = mx'+h, \therefore h = y'-mx';$$

substituting therefore this expression for h in the general equation $y = mx+h$, we get the equation

$$y = mx+y'-mx', \text{ or } y-y' = m(x-x') \quad . \quad . \quad . \quad (1),$$

in which the only arbitrary or disposable constant is m, which may still take any value we may choose to give it.

The equation (1) therefore, by giving various values, one after another, to m, will represent all the innumerable lines which pass through the given point, to the exclusion of all other lines. Hence (1) is *the general equation of a straight line passing through the point* (x', y').

If the given point be on the axis of x, then $y' = 0$; and the equation is

$$y = m(x-x').$$

If it be on the axis of y, then $x' = 0$; and the equation is

$$y - y' = mx, \text{ or } y = mx + y',$$

as we might foresee it must be; since, by the condition, y' is the intercept which the required line must cut off from the axis of y.

Suppose, by way of illustration, that the co-ordinates ot the given point are $x' = 2$, $y' = -3$, in reference to assigned axes; then the equation $y + 3 = m(x - 2)$ represents a straight line passing through the point $(2, -3)$; and a *different* straight line through the point, for every *different* value we give to m.

PROBLEM 2.—*To find the equation of a straight line which passes through* two *given points.*

Let the given points be (x', y'), and (x'', y''). Since *all* the straight lines passing through the first of these points are comprehended in the equation $y - y' = m(x - x')$, we have to select, and specify, that particular line of the group which passes *also* through the second point (x'', y''), and which must therefore satisfy the additional condition.

$$y'' - y' = m(x'' - x') \therefore m = \frac{y'' - y'}{x'' - x'} \text{ or } \frac{y' - y''}{x' - x''}.$$

Substituting this value of m in the above equation, we have for the analytical representation of the single line fulfilling the two proposed conditions,

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') \quad . \quad . \quad . \quad (1),$$

in which all are *assigned constants*, except the *general* variables x and y. The coefficient of inclination m is the given quantity $\frac{y' - y''}{x' - x''}$; and the constant h is the given quantity $\frac{x'y'' - y'x''}{x' - x''}$; so that nothing is left arbitrary.

The student will do well to pay special attention to the analysis of the two foregoing problems, and to observe how, from an expression of the utmost generality, by introducing a restrictive condition, we educe from it a second expression, of limited application; that is, how from an equation representing every straight line that can possibly be drawn in the plane of the axes of reference, we derive another, less general equation, which shall represent only a particular group of these innumerable lines, to the exclusion of all the others; and then, by introducing among these a second restrictive condition, how, from *this* assemblage of lines, that *particular line* is selected which, alone, satisfies *both* conditions.

As an illustration of the present problem, let the coordinates of one of the given points (x', y') be 4 and 6, and those of the other given point (x'', y'') 3 and 5; then

$$m = \frac{y'-y''}{x'-x''} = \frac{1}{1} = 1, \text{ and } h = \frac{x'y''-y'x''}{x'-x''} = \frac{2}{1} = 2,$$

and therefore the equation of the line passing through the two points is

$$y = 1x + 2;$$

so that if the axes of reference are rectangular, 1 is the trigonometrical tangent of the inclination of the line to the axis of x, which inclination is therefore 45°. The ordinate at the origin, that is, the intercept of the axis of y, is 2, and the other intercept is -2. But if the assigned axes be *not* rectangular, then the coefficient 1 expresses the ratio $\frac{\text{AB}}{\text{AC}}$ (see fig. at p. 10): the intercepts however are the same, so that whether the axes are rectangular or oblique, the construction of the line is effected with equal facility.

Without referring to the symbolical expressions for m and h, as above, in any particular case, we may find the numerical values of these quantities in a direct manner: thus, in the case before us, we have to satisfy by suitable value for m and h, the two simultaneous conditions

$$\left.\begin{array}{r} 6 = 4m+h, \text{ or } 4m+h-6 = 0 \\ \text{and } 5 = 3m+h, \text{ or } 3m+h-5 = 0 \end{array}\right\}$$

Subtracting the second of these from the first, we have $m-1=0$, $\therefore\ m = 1$, $\therefore$ (from the second), $3+h-5=0$, $\therefore\ h = 2$.

When one of the given points, as the point (x'', y''), is at the origin, then $x''=0$, and $y''=0$; and the equation of the straight line passing through the two points is simply

$$y=\frac{y'}{x'}x,$$

as we already know from page 10, $\frac{y'}{x'}$ being the same ratio as $\frac{\text{AB}}{\text{AC}}$, wherever on the line the point (x', y') may be, and whatever be the inclination of the axes.

PROBLEM 3.—*To find the equation of the straight line which passes through a given point, and is also parallel to a given straight line.*

Let the given point be (x', y'): we have for the general equation of every line passing through it

$$y-y'=m(x-x'),$$

and among these we have to distinguish that particular one which is parallel to the given line, $y=m'x+h'$. Now two lines are obviously parallel when their coefficients of inclination are equal; therefore, we have only to give the particular value m' to m, in the foregoing equation, and then it will become

$$y-y'=m'(x-x'),$$

which represents the line passing through the point (x',y'), and parallel to the line $y=m'x+h'$.

For example:—Let it be required to find the equation of the straight line passing through the point $(2,-3)$, and parallel to the line $5y-3x+4=0$. Putting this equation in the form

$$y=mx+h, \text{ it is } y=\frac{3}{5}x-\frac{4}{5},$$

$\therefore \frac{3}{5}$ is the coefficient of inclination; and the required equation is therefore $y+3=\frac{3}{5}(x-2)$.

PROBLEM 4.—*The equations of two straight lines being given, to find the co-ordinates of the point in which they intersect.*

Both lines being referred to the same axes, let their equations be

$$\left.\begin{array}{l} y = m'x + h' \\ y = m''x + h'' \end{array}\right\}$$

We link these equations together to imply that they are *simultaneous*, or that the values of x and y are the same in the one as in the other, as they must be at the point where the lines intersect, but at that point *only*.

Hence $m'x + h' = m''x + h''$, $\therefore\ x = \dfrac{h'' - h'}{m' - m''}$, $y = \dfrac{m'h'' - m''h'}{m' - m''}$.

This second fraction may be written

$$m'(h'' - h') + \frac{h'(m' - m'')}{m' - m''} = \frac{m'(h'' - h')}{m' - m''} + h',$$

so that the required co-ordinates are

$$x = \frac{h'' - h'}{m' - m''},\ y = \frac{m'(h'' - h')}{m' - m''} + h'.$$

If we take $m' = m''$, the two lines will be parallel; and these expressions for x and y, the co-ordinates of the point where they intersect, will each be infinite, implying that the lines intersect only at an infinite distance; that is, that they do not intersect at all.

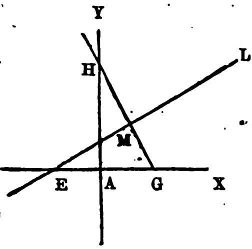
If $h' = h''$, then $x = 0$, and $y = h'$; showing that the lines intersect on the axis of y.

PROBLEM 5.—*To find an expression for the tangent of the angle of intersection of two given straight lines.*

In each of the foregoing problems the axes of reference may be either rectangular or oblique: in the present problem, we shall assume the axes to be rectangular; as *here* it is not a matter of indifference, as it is in the former problems, what angle the axis of y makes with the axis of x.

Let the equations of the two given lines EL, GH, be

$$y = m''x + h'', \text{ and}$$
$$y = m''x - h'' \quad . \quad . \quad . \quad (1)$$



Then since the angle MGX = EMG + MEG (*Euclid*, Prop. 32, Bk. I.), we have EMG = MGX − MEG; or, calling this angle of intersection v, we have $v = a'' - a'$, where a', a'', are put for the angles MEG, MGX, respectively; the tangents of these angles, by (1), being m' and m''.

But (*Trigonometry*, p. 31),

$$\tan(a'' - a'), \text{ or } \tan v = \frac{\tan a'' - \tan a'}{1 + \tan a' \tan a''}$$

$$\text{that is, } \tan v = \frac{m'' - m'}{1 + m'm''} \quad . \quad . \quad . \quad (2).$$

Therefore the given lines intersect at an angle of which the tangent is this expression.

The reciprocal expression for tan v is cot v (*Trigonometry*, p. 15), that is,

$$\cot v = \frac{1 + m'm''}{m'' - m'},$$

so that if $v = 90°$, in which case $\cot v = 0$, we must have the condition $1 + m'm'' = 0$; and therefore,

$$m'm'' = -1, \text{ or } m'' = -\frac{1}{m'}.$$

Hence, if $y = m'x + h'$ be the equation of any given straight line, the equation of any line perpendicular to it will be $y = -\frac{1}{m'}x + h$. There are innumerable perpendiculars to the given line, since h is arbitrary, or open to any interpretation. If one of these perpendiculars be fixed by the condition that it must also pass through a given point (x', y'), then, by Problem 1, its equation will be

$$y - y' = -\frac{1}{m'}(x - x'),$$

a line perpendicular to $y = m'x + h'$, and passing through (x', y').

A perpendicular to $y = m'x + h'$, from the *origin*, at which point $x' = 0$, and $y' = 0$, is characterised by $y = -\frac{1}{m'}x$.

EXAMPLES.

1. Find the equation of the straight line which passes through the point $(-5, -3)$, and is at the same time perpendicular to the line $2y + 3x - 4 = 0$. *Ans.* $y = \frac{2}{3}x + \frac{1}{3}$.

2. Find the point at which the two lines $3y - x + 1 = 0$, and $y + 3x - 1 = 0$, intersect, as also the tangent of the angle of intersection. *Ans.* $x = -\frac{1}{2}$, $y = -\frac{1}{2}$; $\tan = \frac{4}{3}$.

3. Required the equation of the straight line which passes through the point $(-5, -3)$, and is at the same time parallel to the line $2y + 3x - 4 = 0$.

Ans. $y = -\frac{3}{2}x - \frac{21}{2}$.

4. At what angle do the lines $2y + 3x + 1 = 0$, and $2x - 3y + 3 = 0$, intersect? *Ans.* 90°.

5. At what angle are the *two* straight lines represented by the equation $(2y + x)(3y - x) = 0$ inclined to one another? *Ans.* 135°.

NOTE.—As the left-hand member of this equation is composed of two factors, the condition is satisfied in two ways, that is, by each of the two equations

$$2y + x = 0, \text{ and } 3y - x = 0.$$

The proposed equation is therefore the algebraical expression for the *two* straight lines of which these are the equations.

PROBLEM 6.—*To find the equation of the straight line which passes through a given point, and which makes a given angle with a given straight line.*

Let the equation of the given straight line be

$$y = m'x + h';$$

also let (x', y') be the given point, then the equation of the required line will be (Prob. 1),

$$y - y' = m(x - x') \quad . \quad . \quad . \quad (1),$$

the constant m being at present unknown. Putting v for *the tangent* of the given angle, we have (p. 17),

$$v = \frac{m'-m}{1+mm'}, \text{ or } = \frac{m-m'}{1+mm'},$$

$$\therefore\ m = \frac{m'-v}{1+m'v}, \text{ or } = \frac{m'+v}{1-m'v}.$$

Hence the equation (1) of the line to be found is

$$y-y' = \frac{m'-v}{1+m'v}(x-x'), \text{ or } y-y' = \frac{m'+v}{1-m'v}(x-x') \ . \ . \ (2),$$

showing that, in general, two distinct straight lines may be drawn, each making the given angle with the given straight line; a truth which is also obvious from the simplest geometrical considerations: if a perpendicular be drawn from the point to the line, and two points, equidistant from the foot of this perpendicular, be taken upon the line, it is plain that the two lines from the given point to these points will make equal angles with the given line. If, however, the given angle be a *right* angle, then only *one* line from the given point will fulfil the condition. Let us examine how the *two* expressions for m in (2), for this unique case, can become identical.

1st. $\dfrac{m'-v}{1-m'v} = \dfrac{\dfrac{m'}{v}-1}{\dfrac{1}{v}+m'}$: and this, when $v = \infty$, is $\dfrac{-1}{m'}$.

2nd. $\dfrac{m'+v}{1-m'v} = \dfrac{\dfrac{m'}{v}+1}{\dfrac{1}{v}-m'}$; and this, when $v = \infty$, is $\dfrac{1}{-m'}$.

And thus the two lines (2)—which in all other cases are *distinct* lines—in *this* case unite in a single line, viz., the line

$$y-y' = -\frac{1}{m'}(x-x'),$$

which, as we previously knew, passes through the point (x', y'), and is at the same time perpendicular to the given line.

It cannot but be interesting to the student thus to see how completely the analytical conclusions satisfy all the demands of Geometry, even in extreme, and what may at first sight appear to be exceptional, cases. And the like perfect consistency and harmony will be found to prevail throughout ANALYTICAL GEOMETRY.

EXAMPLES.

1. Required the equation of the straight line which passes through the point (2, 3), and which makes an angle whose tangent is 2 with the line $y = 4x + 5$?

Ans. $y-3 = \frac{2}{9}(x-2)$, or $y-3 = -\frac{6}{7}(x-2)$.

2. Required the equation of the straight line which passes through the point $(-2, -3)$, and which makes an angle whose tangent is $\frac{3}{2}$ with the line $y = -4x + h$?

Ans. $y-y' = \frac{11}{10}(x-x')$, or $y-y' = -\frac{5}{14}(x-x')$.

3. Required the equation of the straight line which passes through the point $(2, -4)$, and which makes an angle of 60° with the line $3y - 5x + h$?

$$Ans.\ y-y' = \frac{5-3\sqrt{3}}{3+5\sqrt{3}}(x-x'),$$

$$\text{or } y-y' = \frac{5+3\sqrt{3}}{3-5\sqrt{3}}(x-x').$$

4. Prove that the equation

$$2y^2 - 3xy - 2x^2 - y + 2x = 0$$

represents two straight lines which are perpendicular to each other; and that one of them passes through the point (1, 2), and the other through the point $(2, -\frac{1}{2})$.

PROBLEM 7.—*To find the analytical expression for the length of the perpendicular from a given point to a given straight line.*

Let LN be the given line, and MH the perpendicular upon it from the given point M.

Let the equation of the given line from L be

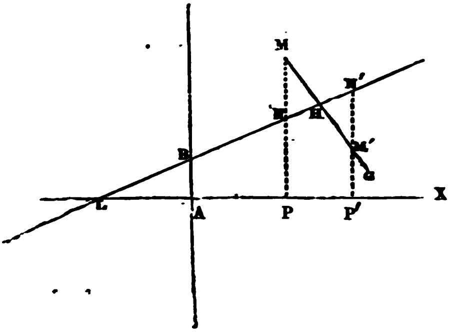
$$y = m'x + h' \quad . \quad . \quad . \quad (1),$$

then the equation of the perpendicular upon it from the point (x', y') or M, will be (Prob. 6).

$$y - y' = -\frac{1}{m'}(x - x') \quad . \quad . \quad . \quad (2).$$

Also the distance D of the point M, from any point (x, y) in the line LN, will be expressed by

$$D = \sqrt{\{(x-x')^2+(y-y')^2\}} \quad . \quad . \quad . \quad (3).$$



Now when H, the foot of the perpendicular, is the point in the line in which alone we are concerned, x and y will have the same values in (1), (2), and (3).

Let us then ascertain the values of $x-x'$ and $y-y'$, from the simultaneous equations (1) and (2); we may then substitute them in the expression (3), and thus obtain for D the particular value MH. The easiest way of finding $x-x'$, and $y-y'$, is this. Put the equation (1) in the form

$$y-y' = m'(x-x')-y'+m'x'+h' \quad . \quad . \quad (4),$$

to which it is evidently equivalent; the equations (2) and (4) will thus both involve the unknown quantities $x-x'$, and $y-y'$; so that their determination will be readily effected; thus, subtract (2) from (4), and there results

$$0 = \left(m'+\frac{1}{m'}\right)(x-x')-y'+m'x'+h',$$

$$\therefore \; x-x' = \frac{m'(y'-m'x'-h')}{m'^2+1},$$

and by substituting this expression for $x-x'$, in the equation (2), we have for $y-y'$,

$$y-y' = -\frac{y'-m'x'-h'}{m'^2+1},$$

and therefore for the distance (3), which distance, being the perpendicular MH, we shall call P, we have

$$\mathrm{P}=\frac{\sqrt{\{(m'^2+1)(y'-m'x'-h')^2\}}}{m'^2+1}$$

$$=\frac{y'-m'x'-h'}{\sqrt{(m'^2+1)}} \quad . \quad . \quad (4).$$

The constant m' is the tangent of the angle a which the given line (1) makes with the axis of x; and we know (*Trigonometry*, p. 9), that $\tan a=\frac{\sin a}{\cos a}$, and that $\sqrt{(\tan^2 a+1)}=\sec a=\frac{1}{\cos a}$; hence, by substitution, the foregoing expression for P becomes

$$\mathrm{P}=y'\cos a-x'\sin a-h'\cos a \quad . \quad . \quad (5),$$

a form of expression for the perpendicular from a point (x', y'), upon the line (1), that is, upon the line

$$y=x\tan a+h',$$

which, in certain investigations, is to be preferred.

If the point (x', y') be the origin, the expression (4) will become

$$\mathrm{P}=\frac{-h'}{\sqrt{(m'^2+1)}}, \text{ and the expression (5),}$$

$$\mathrm{P}=-h'\cos a \quad . \quad . \quad (6).$$

Either of the expressions (4), (6) may be written down at once in any particular case without any effort of memory; for the equation of the given line being put in the form

$$y-m'x-h'=0,$$

the left-hand member of the equation is always the numerator of the fraction P, when the co-ordinates of the given point (x', y') are put for x, y; and the denominator is always $\sqrt{(m'^2+1)}$; that is the secant of the angle a at which the given line is inclined to the axis of x.

If the general equation of a straight line be written

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0, \text{ which gives } m=-\frac{\mathrm{A}}{\mathrm{B}}, \text{ and } h=-\frac{\mathrm{C}}{\mathrm{B}},$$

the value of P takes the form

$$P=\frac{\mathrm{A}x'+\mathrm{B}y'+\mathrm{C}}{\sqrt{(\mathrm{A}^2+\mathrm{B}^2)}} \quad . \quad . \quad . \quad (7),$$

which, when (x', y') is the origin, is simply

$$\text{P} = \frac{\text{C}}{\sqrt{(\text{A}^2+\text{B}^2)}} \quad . \quad . \quad . \quad (8).$$

EXAMPLES.

1. Required, the length of the perpendicular from the point (2, 3), upon the straight line $2y-5x+7=0$?

In this particular case, the expression for P, marked (7), is

$$\text{P} = \frac{2y'-5x'+7}{\sqrt{(2^2+5^2)}} = \frac{6-10+7}{\sqrt{29}} = \frac{3}{\sqrt{29}}.$$

2. Required, the length of the perpendicular from the point (3, 2), upon the line $5y-2x=0$? *Ans.* $\text{P} = \frac{4}{\sqrt{29}}$.

3. Required, the length of the perpendicular from the point $(-3, 5)$, upon the line $3y+4x=12$? *Ans.* $\text{P} = -\frac{9}{5}$.

4. Required, the length of the perpendicular from the point (3, 5), upon the line $7x-3y-9=0$?

Ans. $\text{P} = -\frac{3}{\sqrt{58}}$.

NOTE.—The minus sign in this, and in the answer to the last Example, may be suppressed; the *length* of the perpendicular has nothing to do with the *sign* which the algebra prefixes to it. The student will find an explanation of the cause of the entrance of this superfluous sign at the end of the present section.

5. Required, the length of the perpendicular from the point $(5, -2)$ upon the line $y = x \tan 20° + 3$?

Ans. $\text{P} = 5(\cos 20° + \sin 20°)$.

6. Required, the length of the perpendicular from the point (0, 0) upon the line $y = \frac{1}{3}x\sqrt{3}-2$; or, which is the same thing, upon the line $y = x \tan 30° - 2$?

Ans. $\text{P} = \sqrt{3}$, or $\text{P} = 2 \cos 30°$.

PROBLEM 8.—*To find the equation of the straight line which bisects the angle between two given straight lines.*

Every point in the straight line which bisects an angle is evidently such that the two perpendiculars drawn from it to the two sides of the angle must be *equal*, and that they cannot be equal except the point from which they are drawn be in the bisecting line. Hence, if (x, y) denote any point indifferently in the bisector, and we write first the expression for the perpendicular from it to one of the two sides of the angle, and then the expression for the perpendicular upon the other side, the two expressions will be *equal*; that is, we shall have an equation in which x and y will be the only unknown quantities, and which will therefore exhibit the relation between these two variables; in other words, we shall get the equation of the required line.

EXAMPLES.

1. Required, the equation of the straight line which bisects the angle between the lines whose equations are

$5y-2x=0$ . . (1), and $3y+4x-12=0$ . . (2).

1st. The perpendicular from any point (X, Y), upon (1), is

$$P=\frac{5Y-2X}{\sqrt{29}},$$

and the perpendicular from the same point upon (2), is

$$P'=\frac{4X+3Y-12}{5}.$$

But if (X, Y) be a point in the line bisecting the angle, then $P=P'$, that is, putting now the symbols x, y, for X, Y,

$$\frac{5y-2x}{\sqrt{29}}=\frac{4x+3y-12}{5},$$

or $(25-3\sqrt{29})y-(10+4\sqrt{29})x+12\sqrt{29}=0$; that is,

$$8{\cdot}86y-31{\cdot}52x+64{\cdot}56=0,$$

the equation of the bisector of the angle.

It will be convenient here to define the term *locus*. The *locus* of a point varying in position is the line, whether straight or curve, in which the point, in all its positions, is *placed*; so that when we enquire what is the locus of a point satisfying certain conditions, we ask for the determination of the *line* which the varying point traces out. The present problem might be otherwise proposed in the following terms, namely: From a point between two given straight lines a perpendicular to each line is drawn; these *perpendiculars* are equal; what is the locus of the point?

The answer would be, that the locus is a straight line passing through the point of junction of the two given lines, if they meet; and parallel to them if they themselves are parallels: it is an obvious inference from the fact that the locus is a *straight* line, and that each point in it is equidistant from the given lines, that it bisects the angle between the sides of which every point in the line is posited.

[The student is recommended to construct, at least roughly, the lines in each of the following examples, taking any convenient length he pleases for the unit of measure; and to notice especially between *which* of the two adjacent angles formed by the given lines the bisector is drawn.]

2. Required, the equation of the straight line which bisects the angle between the lines $4y = 5x+12$, and $5y-4x+1 = 0$? *Ans.* $y+x+13 = 0$.

3. Required, the equation of the straight line which bisects the angle between the lines $3y+2x-7 = 0$, and $3x-2y+9 = 0$? *Ans.* $5y-x-16 = 0$.

4. Required, the equation of the straight line which bisects the angle between the lines $2y-\frac{3}{2}x+4 = 0$, and $y+3x-6 = 0$?

Ans. $(4\sqrt{10}-5)y-3(5+\sqrt{10})x+30+8\sqrt{10} = 0$.

5. Required, the equation of the straight line which bisects the angle between the lines $2y+x-5 = 0$, and $3y-x+8 = 0$?

Ans. $(2\sqrt{2}-3)y+(1+\sqrt{2})x-(8+5\sqrt{2}) = 0$.

It may be well to notice here that when the equations of two given straight lines, written in the form

$$\text{A}x+\text{B}y+\text{C} = 0,\quad \text{A}'x+\text{B}'y+\text{C}' = 0, \quad . \quad . \quad . \quad (1)$$

are either added together, or one subtracted from the other, the resulting equation

$$(\text{A}x+\text{B}y+\text{C}) \pm (\text{A}'x+\text{B}'y+\text{C}') = 0 \quad . \quad . \quad . \quad (2)$$

is always the equation of some straight line passing through their point of intersection, whenever the lines *do* intersect. For the equation is that of *some* straight line, because it is of the first degree; and it passes through the point of intersection of the given lines, because it is satisfied for the pair of values for x, y, which cause the two compound terms of the equation to vanish simultaneously. The equation of the line which is the answer to Example 2, is the *difference* of the equations of the given lines; and in this case, the line passing through the intersection of the given lines turns out to be the *bisector* of the angle formed by them.

What is here said, applies, of course, when one or both of the compound terms of (2) are multiplied by any constant factor; thus, the equation

$$a(\mathrm{A}x+\mathrm{B}y+\mathrm{C})\pm b(\mathrm{A}'x+\mathrm{B}'y+\mathrm{C}')=0 \quad . \quad . \quad . \quad (3)$$

represents a straight line (or rather *two* straight lines) meeting the lines (1) at their point of intersection; and if the constants a, b, be

$$a=\frac{1}{\sqrt{(\mathrm{A}^2+\mathrm{B}^2)}},\quad b=\frac{1}{\sqrt{(\mathrm{A}'^2+\mathrm{B}'^2)}},$$

then the two adjacent angles, one the supplement of the other, formed by the two lines (1), will be bisected by the two lines (3).

On the sign of the perpendicular from a given point upon a given straight line.

In the original conventions of Analytical Geometry, no special provision is made for the directive signs of straight lines oblique to the co-ordinates; and when the *length* only of any line is to be determined, a sign, prefixed to the algebraical expression for such length, is, of course, superfluous: yet when the algebra furnishes such a prefix, it will always be found to do so as a necessary consequence of the co-ordinate conventions. The sign prefixed to the length of the perpendicular from a given point to a given straight line, is an illustration of this: thus, suppose the line LN, in the diagram at p. 21, to turn about the point L, till it come into the position of parallelism with the axis of y: the perpendicular upon it from A will then be AL; which, being in a co-ordinate direction, will, of course take the sign *minus.* This, however, it would not do, but for the *minus* prefixed to the *general* expression (6), for a perpendicular upon *any* straight line from A; since throughout the revolution of the line LN (making an *acute* angle α with LX) the sign of $\cos\alpha$ is +, and that of h', or AB, is also +, however remote the point B may be.

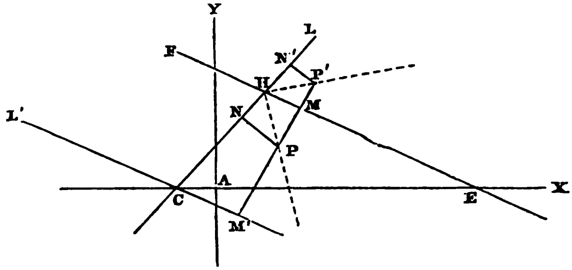
If we refer to the diagram immediately before the one here considered (p. 17), and regard HG as the given line, the angle a is *obtuse*; and if the line turn about H, till it come into the extreme position of parallelism with the axis of x, the perpendicular upon it from A will then be AH; which, being in a co-ordinate direction, will take the proper sign *plus*: this, however, it would not do, but for the *minus* prefixed to the *general* expression (6), in which cos a is negative throughout all the successive positions of the revolving line up to the extreme limit; and h', or AH, is *positive*. To these remarks we may further add:

1. The sign of the denominator, in the expression marked (4) at p. 22, is never *ambiguous*, for that denominator is the *secant* of the fixed angle a which the line makes towards the right with the axis of x. Its sign is that of m'.

2. The sign of the numerator is + or −, according to the relative positions of the line and the point: here, too, there is no ambiguity of sign; one or other appears of necessity, as an inevitable consequence of the co-ordinate conventions. If the line be moved in its own direction, and then parallel to itself, carrying the perpendicular with it, till the point coincides with the origin, the proper sign will be ascertained from the expression (6) already referred to; and we thus see how improper it is (as is, however, frequently done) to prefix the *ambiguous* sign to the expression (4), when the point and the line are both *fixed*.

3. When, however, the point is *not* fixed, that is, when we do not know whether it be situated on one side of the given line, or on the other, for *two* points may be equidistant from a straight line lying between them, then indeed the sign of the distance *is* ambiguous; if the sign for one of the perpendiculars be *plus*, the sign for the opposite perpendicular, in virtue of this opposition, must be *minus*; but which of the two is entitled to the *plus*, and which to the *minus*, can be ascertained only by reference to one or other of the general formulas above alluded to; or by imagining the given line to revolve into a position of parallelism with one of the co-ordinate axes, in the manner already explained.

Let us now consider the cases coming under Problem 8, the cases, namely, in which two perpendiculars, of equal lengths, are to be drawn from a point (undetermined) upon two straight lines meeting in an angle: the two straight lines CL, EF, for instance, meeting in the point H.



Let P be one of the points in the line of points, all lying *within* the angle CHE, such that PM = PN; and P′ one of the points within the supplemental angle EHL, such that P′M = P′N′, each point fulfilling the proposed conditions; then the locus of P is the straight line bisecting the angle CHE, and the locus of P′, the straight line bisecting the angle EHL. Conceive CL to turn about C till it comes into parallelism with EF: then the perpendiculars upon the two lines EF, CL′ from P will be PM, PM′ lying in *opposite* directions, and therefore claiming opposite signs; but the perpendiculars from P′, namely, P′M, P′M′, lie in *the same* direction, and therefore claim the same signs.

In the example worked at p. 24, we have assumed the two perpendiculars, there represented by P, P′, to have *the same* signs; that is, to be either both plus, or both minus. In consequence of this assumption the line which the analysis determines is the bisector, not of the angle CHE, but of the exterior angle EHL;* and in each of the examples which follow this, the circumstances are the same. Let us, however, now take the expressions for P and P′ (p. 24), with *opposite* signs; that is, let us take

$$P = \frac{2x - 5y}{\sqrt{29}}, \text{ and } P' = \frac{4x + 3y - 12}{5},$$

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* The two intersecting lines CL, EF, in the diagram, are drawn arbitrarily, and are not adapted to any particular example.

then the equation $P = P'$, will be found to give

$$41{\cdot}14y + 11{\cdot}52x - 64{\cdot}56 = 0;$$

for the equation of the straight line which bisects the angle CHE.

We may verify this as follows; making, of course, due allowance for the very limited number of decimals (only two places) to which the coefficients have been computed.

The equation of the external bisector, at p. 24, is

$$y = \frac{31{\cdot}52}{8{\cdot}86}x - \frac{64{\cdot}56}{8{\cdot}86} \quad . \quad . \quad . \quad (1),$$

and that of the line here affirmed to be the internal bisector, is

$$y = -\frac{11{\cdot}52}{41{\cdot}14}x + \frac{64{\cdot}56}{41{\cdot}14} \quad . \quad . \quad . \quad (2).$$

Now $\frac{8{\cdot}86}{31{\cdot}52} = {\cdot}28 \ldots$, and $\frac{11{\cdot}52}{41{\cdot}14} = {\cdot}28 \ldots$

Consequently the lines (1) and (2) are perpendicular to each other (p. 17), so that if one of them bisect either angle, the other must bisect the adjacent, or supplementary angle; for the two bisectors are necessarily the one perpendicular to the other.

If we take Example 2 (p. 25), and treat it as above, we shall have, for the equation of the line,

$$y - x - \frac{11}{9} = 0,$$

that of the line

$$y + x + 13 = 0,$$

in the answer to that Example, being perpendicular to it, as it ought to be. And similarly of the other examples.*

We shall now apply the principles established in this and in the preceding sections to a few problems of geometry in which intersecting straight lines are concerned.

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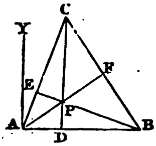
* The foregoing discussion, which the editor believes to have som claim to novelty, cannot fail to be satisfactory and instructive to the enquiring student.

Section IV.—PROBLEMS CONCERNING THE INTERSECTIONS OF STRAIGHT LINES, &c.

Problem 1.—*To determine whether or not the perpendiculars drawn from the three vertices of a plane triangle to the opposite sides, all intersect in one point.*



Let ABC be any plane triangle, and CD, AF, BE, the three perpendiculars to the sides opposite to C, A, B, respectively.

Take the base AB for the axis of x, and AY, the perpendicular to it, for the axis of y. Put x' y' for the co-ordinates AD, DC, of the point C; also for AB, the abscissa of the point B, put x'': then the analytical representations of the three points will be

for A, $(0, 0)$; for B, $(0, x'')$; and for C, (x', y');

and our object is to ascertain whether the abscissa of the point of intersection of AF, BE, is also the abscissa of the point C; that is, whether the x of this intersection is x'.

The straight lines concerned in this enquiry are these, namely,

AC, passing through the two points $(0, 0)$, and $(x'\ y')$.
BC, passing through the two points $(x'', 0)$, and (x', y').
BE, passing through the point $(x'', 0)$, and perp. to AC.
AF, passing through the point $(0, 0)$, and perp. to BC.

The equations of these four lines are as follows, namely:

$$\text{Equation of AC, } y = \frac{y'}{x'}\, x;$$

$$\text{Equation of BC, } y = \frac{y'}{x'-x''}\,(x-x'');$$

$$\text{Equation of BE, } y = -\frac{x'}{y'}\,(x-x'');$$

$$\text{Equation of AF, } y = -\frac{x'-x''}{y'}\, x.$$

Now at the point where AF, BE, intersect, the ordinates must be the same for both lines: hence at this point, we must have

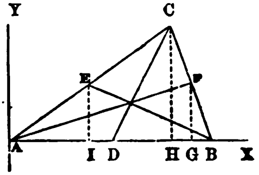
$$\frac{x'}{y'}(x-x'') = \frac{x-x''}{y'}x, \therefore x'(x-x'') = (x'-x'')x \therefore x = x'.$$

Consequently the abscissa AD of the point P, in which the lines AF, BE, intersect, is also the abscissa of the point C; that is, the perpendicular CD passes through the point P, at which point therefore all the three perpendiculars meet.

PROBLEM 2.—*To determine whether or not the straight lines drawn from the three vertices of a triangle to bisect the opposite sides meet in the same point.*

Let the axis of x coincide with the base AB of the triangle, the axis of y, being drawn from A perpendicular to AB; let AB be called c: then putting x', y', for the co-ordinates of C, the three points A, B, C, will be analytically expressed thus:

A, $(0, 0)$; B, $(c, 0)$; C, (x', y'); and since D, E, F, are the middle points of AB, AC, BC, we shall have



$$AD = \frac{c}{2};\quad EI = \frac{y'}{2};\quad AI = \frac{x'}{2};$$

$$FG = \frac{y'}{2};\quad HG = \frac{HB}{2} = \frac{c-x'}{2},$$

therefore, $AG = 2AI + HG = x' + \frac{c-x'}{2} = \frac{c+x'}{2}$; so that we have for the points D, E, F, the following analytical expressions:

$$D, \left(\frac{c}{2}, 0\right);\quad E, \left(\frac{x'}{2}, \frac{y'}{2}\right);\quad F, \left(\frac{c+x'}{2}, \frac{y'}{2}\right).$$

Now each of the lines AF, BE, CD, passes through two fixed points, the co-ordinates of which are stated above; therefore the equation of each line is of the form

$$y - y' = \frac{y'-y''}{x'-x''}(x-x'), \text{ p. } 13,$$

and substituting in this equation the particular values for x', y'; x'', y'', already determined, we have

FIRST; for AF, the equation $y-0=\dfrac{0-\frac{y'}{2}}{0-\frac{c+x'}{2}}(x-0)$;

that is, $y=\dfrac{y'}{c+x'}x$, the equation of AF . . . (1).

SECOND; for BE, the general form above becomes

$$y-0=\frac{0-\frac{y'}{2}}{c-\frac{x'}{2}}(x-c),$$

that is, $y=\dfrac{y'}{-2c+x'}(x-c)$, the equation of BE . . . (2).

THIRD; for CD, the general equation gives

$$y-y'=\frac{y'}{x'-\frac{c}{2}}(x-x'),$$

$$\text{that is, } y=\frac{2x'y'-cy'+2y'x-2x'y'}{2x'-c};$$

$\therefore y=\dfrac{y'}{2x'-c}(2x-c)$, the equation of CD . . . (3).

Now, if these three lines (1), (2), and (3), all meet in the same point (x, y), there must be values for these co-ordinates which will satisfy all three of the equations, but not else. Equating (1) and (2), we have

$$\frac{y'}{c+x'}x=\frac{y'}{x'-2c}(x-c);$$

$$\therefore 2x'y'x-cy'x=2cy'x-c^2y'+2x'y'x-cx'y',$$

that is, $3cy'x=cy'(c+x')$ $\therefore x=\dfrac{c+x'}{3}$, and this value of x, substituted in the equation (1), gives for the corresponding value of $y, y=\dfrac{y'}{3}$. Hence the co-ordinates of the point in *which the* lines AF, BE, intersect are

$$x=\frac{c+x'}{3},\ \text{and}\ y=\frac{y'}{3}.$$

Again: equating (1) and (3), we have

$$\frac{y'}{c+x'}\,x=\frac{y'}{2x'-c}\,(2x-c);$$

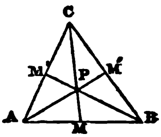
$$\therefore\ 2x'y'x-cy'x=2cy'x-c^2y'+2x'y'x-cx'y';$$

$$\therefore\ c^2y'+cx'y'=3cy'x,\ \therefore\frac{c+x'}{3}=x,\ \text{and}\ \therefore\ y=\frac{y'}{3};$$

so that the co-ordinates of the point in which AF, BE, meet, are the same as the co-ordinates of the point in which AF, CD, meet; that is, all three of the bisecting lines meet in the same point.

Second Method of Investigation.

Let MC, which bisects AB, be taken for the axis of y, and MB for the axis of x; then M′, M″, being the middle points of AC, BC, a parallel to CM, from M′, will bisect AM, and be equal to half CM. In like manner, a parallel to CM, from M″, will bisect MB, and this will also be equal to half CM: hence, if $(x', 0)$ denote the point B, the point A will be $(-x', 0)$; and if (x'', y'') denote the point M″, the point M′ will be $(-x'', y'')$: therefore, the points concerned will be analytically denoted thus:



A, $(-x', 0)$; B, $(x', 0)$; M′, $(-x'', y'')$; M″, (x'', y''),
and consequently the equations of AM″, BM′, will be

$$y=\frac{y''}{x''+x'}\,(x+x'),\ \text{the equation of AM}''\quad .\ .\ .\ (1),$$

$$y=\frac{y''}{-x''-x'}\,(x-x'),\ \text{the equation of BM}'\quad .\ .\ .\ (2).$$

Now, if these two lines intersect on MC, the axis of y, they must have a common ordinate, MP, at the origin; in other words, (1) and (2) must give the same value of y for the value $x=0$; if such be *not* the case, the two lines cannot have their intersection on the axis of y. Putting $x=0$ in

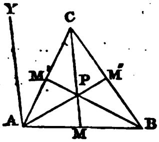
(1), there results $y = \frac{y''}{x''+x'}\, x'$; and putting $x = 0$ in (2),

$$\text{there results } y = \frac{y''}{-x''-x'}\,(-x') = \frac{y''}{x''+x'}\, x';$$

therefore the ordinate of the point where one of the two lines AM″, BM′, crosses CM, is the same as the ordinate of the point where the other crosses CM: hence the three bisecting lines all meet in the same point P; the ordinate PM being,

$$\text{PM} = \frac{y''}{x''+x'}\, x' = \frac{\frac{1}{2}\text{CM}}{\frac{3}{2}x'}\, x' = \tfrac{1}{3}\text{CM}.$$

Third Method of Investigation.



Let the axes now be AB, AY, the latter being parallel to MC; also let C be (x', y'), then B will be $(2x', 0)$; M′; $(\frac{1}{2}x', \frac{1}{2}y')$, and M″; $(x'+\frac{1}{2}x', \frac{1}{2}y')$ and we have to ascertain whether the straight line through A, $(0, 0)$, and the second of these, and that through B, $(2x', 0)$, and the first, intersect in a point of which the abscissa is $x = x'$.

The equation of AM″ is $y = \frac{\frac{1}{2}y'}{\frac{3}{2}x'}x$, and the equation of BM′

$$\text{is } y = \frac{-\frac{1}{2}y'}{2x'-\frac{1}{2}x'}\,(x-2x').$$

At the point of intersection, these equations are simultaneous,

$$\therefore \frac{x}{\frac{3}{2}} = \frac{2x'-x'}{\frac{3}{2}}, \therefore x = x'.$$

Hence the two lines AM″, BM′, intersect on CM.

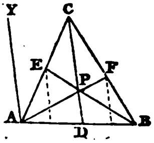
The y of this intersection is $y = \frac{\frac{1}{2}y'}{\frac{3}{2}x'}x' = \frac{1}{3}y'$; that is,

$$\text{PM} = \tfrac{1}{3}\,\text{CM}.$$

Problem 3.—*To determine whether or not the straight lines bisecting the three angles of a triangle meet in a point.*

Let the straight lines AF, BE, bisect the two angles A, B of the triangle ABC; the question is, do these lines intersect each other on CD, the line which bisects the angle C?

Take DC, DB, for axes of co-ordinates, and let P be the point in DC where the line AF crosses DC; and if possible let BE cross DC in some other point P′*; then the equations of AF, BE are (p. 8)



$$\mathrm{AF},\ \frac{x}{-\mathrm{DA}}+\frac{y}{\mathrm{DP}}=1;\ \mathrm{BE},\ \frac{x}{\mathrm{DB}}+\frac{y}{\mathrm{DP}'}=1.$$

The ordinates of the points where these lines cross the axis of y, that is, DC, are $y=\mathrm{DP}$, and $y=\mathrm{DP}'$, and we have now to ascertain whether these ordinates are *equal.*

By Trigonometry

$$\frac{\mathrm{PD}}{\mathrm{AD}}=\frac{\sin\frac{1}{2}\mathrm{A}}{\sin\frac{1}{2}(\mathrm{A}+\mathrm{C})}=\frac{\sin\frac{1}{2}\mathrm{A}}{\cos\frac{1}{2}\mathrm{B}};$$

$$\frac{\mathrm{P}'\mathrm{D}}{\mathrm{DB}}=\frac{\sin\frac{1}{2}\mathrm{B}}{\sin\frac{1}{2}(\mathrm{B}+\mathrm{C})}=\frac{\sin\frac{1}{2}\mathrm{B}}{\cos\frac{1}{2}\mathrm{A}};$$

$$\therefore\ \frac{\mathrm{PD}}{\mathrm{P}'\mathrm{D}}=\frac{\sin\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{A}}{\sin\frac{1}{2}\mathrm{B}\cos\frac{1}{2}\mathrm{B}}\cdot\frac{\mathrm{DB}}{\mathrm{AD}}=\frac{\sin\mathrm{A}}{\sin\mathrm{B}}\cdot\frac{\sin\mathrm{B}}{\sin\mathrm{A}}\ (\textit{Euclid}, 3, \text{VI.});$$

$$\therefore\ \frac{\mathrm{PD}}{\mathrm{P}'\mathrm{D}}=1,\ \therefore\ \mathrm{PD}=\mathrm{P}'\mathrm{D}.$$

Hence P and P′ are one and the same point on CD.

Second Method of Investigation.

The same oblique axes of reference being taken as above, we have $-\mathrm{DA}$ for the abscissa of the point A, and DB for the abscissa of the point B; consequently the equations of AF, BE, may be put in the following forms (p. 8), namely,

Equation of AF, $y=m(x+\mathrm{AD})$;
equation of BE, $y=m_1(x-\mathrm{DB})$.

The coefficient m, as shown at p. 10, is the ratio

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* This hypothetical point P′ is not in the diagram: it is proved that P and P′ are *identical.*

$$m = \frac{\sin \text{FAB}}{\sin(\text{CDB}-\text{FAB})} = \frac{\sin \frac{1}{2}\text{A}}{\sin \frac{1}{2}(\text{A}+\text{C})} = \frac{\sin \frac{1}{2}\text{A}}{\cos \frac{1}{2}\text{B}}.$$

And in like manner,

$$m_1 = -\frac{\sin \frac{1}{2}\text{B}}{\cos \frac{1}{2}\text{A}}.$$

If these values of m, and m_1, be substituted in the equations of AF and BE just given, and then 0, the abscissa of the origin, be put for x in each equation, the resulting ordinates at the origin will be

$$\text{for AF, } y = \frac{\sin \frac{1}{2}\text{A}}{\cos \frac{1}{2}\text{B}}\text{AD}; \text{ and for BE, } y = \frac{\sin \frac{1}{2}\text{B}}{\cos \frac{1}{2}\text{A}}\text{DB};$$

and it now remains to be ascertained whether or not these two expressions are identical.

In order to this, substitute for AD its equal, namely,

$$\text{AD} = \frac{\sin \text{B}}{\sin \text{A}}\text{DB} = \frac{\sin \frac{1}{2}\text{B} \cos \frac{1}{2}\text{B}}{\sin \frac{1}{2}\text{A} \cos \frac{1}{2}\text{A}}\text{DB}$$

(see *Trigonometry*, p. 27);

and the first of the above expressions for y becomes

$$y = \frac{\sin \frac{1}{2}\text{B}}{\cos \frac{1}{2}\text{A}}\text{DB},$$

which is identical with the second expression for y; hence the three lines AF, BE, CD, all meet in the same point P.

From the careful study of these worked-out examples the learner will no doubt be able to investigate the following for himself.

PROBLEMS FOR EXERCISE.

1. It is required to prove that the perpendiculars, drawn from the middle points of the three sides of a plane triangle, all meet in the same point.

2. A line is drawn parallel to the base of a triangle, and from the extremities of this line, lines crossing each other in a point P, are drawn to the extremities of the base. Prove that the locus of P is a straight line from the vertex to the middle of the base.

3. In any parallelogram let a diagonal be drawn and prolonged; let also a diagonal of each of the parallelograms about this diagonal be drawn and prolonged. Prove that the three lines thus drawn all meet in the same point.

4. If (x', y'), (x'', y''), (x''', y'''), be three points in the same straight line, prove that the co-ordinates must satisfy the following condition, namely,

$$(x'y''-x''y')+(x''y'''-x'''y'')+(x'''y'-x'y''') = 0.$$

5. It is required to assign the relation amongst the coefficients in the three equations

$$y = m'x+h', \ y = m''x+h'', \ y = m'''x+h''',$$

so that the lines represented by them may all pass through the same point.

The relation is

$$(m'm''-m''m')+(m''m'''-m'''m'')+(m'''m'-m'm''') = 0.$$

6. It is required to express the area of a triangle in terms of the rectangular co-ordinates of its three vertices, these vertices being the points (x_1, y_1), (x_2, y_2), (x_3, y_3).

The expression for the area is

$$\text{Area} = \tfrac{1}{2} \{(x_2-x_3)y_1+(x_3-x_1)y_2+(x_1-x_2)y_3\}.$$

And if one of the vertices, (x_3, y_3), be at the origin, the expression is,

$$\text{Area} = \tfrac{1}{2}(x_2y_1-x_1y_2).$$

SECTION V.—ON THE CIRCLE: THE EQUATION OF ITS CIRCUMFERENCE, ETC.

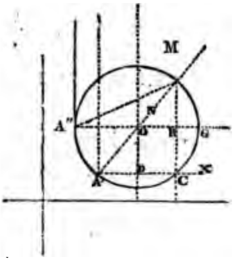
Let r be the radius of any circle of which the centre is O; and let Ax, Ay, be any pair of co-ordinate axes.

Then if (α, β) denote the centre O, and (x, y) any point M in the circumference, the distance OM, between the points (α, β), and (x, y) will be (p. 4)

$$(x-\alpha)^2+(y-\beta)^2 = r^2 \quad . \quad . \quad . \quad (1),$$

an equation which applies to every point in the circumference, and is therefore called *the general equation of the circle in reference to rectangular axes.*

That the equation is satisfied *only* for points in the circumference, and can never denote a point out of the circumference may be easily shown. Let N be any point taken either within or without the circumference; then the distance ON between (α, β) and this point (x, y) is



$$(x-\alpha)^2+(y-\beta)^2 = \text{ON}^2.$$

But if the point N be within the circumference, ON will be less than OM or r; and if it be without the circumference, ON will be greater than OM or r, so that the equation (1) cannot be satisfied unless the point (x, y) is *on* the circumference of the circle whose radius is r, and centre (α, β); and these constants α, β, and r, being given or fixed, the individual circle becomes fixed both in magnitude and situation; and points in its circumference, as many as we please, may be determined from the equation (1).

This general equation of the circle leaves the origin of the rectangular axes quite unrestricted as to position; but if the origin be a point on the circumference, the equation takes a simpler form: thus, suppose the origin to be at A′, then, since in this case $\alpha^2+\beta^2$, that is $\text{A}'\text{D}^2+\text{DO}^2 = r^2$, the general form $(x-\alpha)^2+(y-\beta)^2 = r^2$, or $x^2-2\alpha x+\alpha^2+y^2-2\beta y+\beta^2 = r^2$, becomes

$$x^2-2\alpha x+y^2-2\beta y = 0, \text{ or } x^2+y^2-2(\alpha x+\beta y) = 0 \quad . \; . \; (2),$$

the equation of the circle when the origin of the rectangular axes is on the circumference. If under this condition as to the origin, the axis of x (A″G), pass through the centre, in which case $\alpha = r$, and $\beta = 0$, the form becomes still more simple; it is then

$$x^2+y^2-2rx = 0, \text{ or } y^2 = (2r-x)x \quad . \; . \; . \quad (3),$$

the equation of the circle when the origin is on the circumference and the axis of x passes through the centre.

But if the axis of y pass through the centre, the form is

$$x^2+y^2-2ry = 0, \text{ or } x^2 = (2r-y)y \quad . \; . \; . \quad (4),$$

since then $\alpha = 0$, and $\beta = r$.

The simplest form of all is when the centre itself is the origin, for then α and β *both* disappear from the equation, which becomes reduced to

$$x^2+y^2 = r^2 \quad . \; . \; . \quad (5)$$

the equation of the circle when the origin of the rectangular axis is at the centre; and on account of its superior simplicity, this is the form in which the equation of the circle is usually expressed when we are at liberty to take any point we please for the origin of the axes.

From the equation (3) a geometrical property of the circle at once suggests itself; for that equation, being convertible into the proportion

$$x : y :: y : 2r-x,$$

that is, $A''R : RM :: RM : RG$,

we infer that a perpendicular MR from any point in the circumference of a circle to a diameter A″G, is a mean proportional between the two parts A″R, RG, into which the perpendicular divides the diameter.

By writing the equation just referred to thus: $x^2+y^2 = 2rx$, another property may be readily deduced as follows: Draw the chord A″M, then the interpretation of the first member of this equation is $A''R^2+RM^2$ which is equal to $A''M^2$; and that of the second member is $A''G \times A''R$, so that $A''G \times A''R = A''M^2$; that is,

$$A''G : A''M :: A''M : A''R,$$

showing that a chord drawn from one extremity of a diameter is a mean proportional between the diameter itself and that remote part of it which the perpendicular from the extremity of the chord cuts off.

Equation of the circle referred to oblique axes.

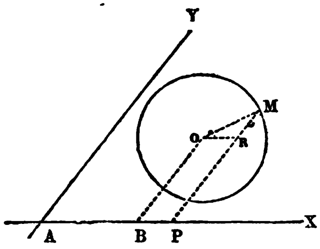
When the angle A is an oblique angle instead of a right angle, the equation of the circle is less simple; we arrive at it thus:

Putting, as before, α, β, for the co-ordinates of the centre O, r for the radius, and (x, y) for any point M in the circumference, we have (page 5) for the distance OM, that is, for r,

$$(x-\alpha)^2+(y-\beta)^2+2(x-\alpha)(y-\beta)\cos A = r^2 \quad . \quad . \quad (6)$$

which is the general equation of the circle when the axes of reference are oblique; it reduces to the general equation for rectangular axes when $\cos A = 0$, as it ought to do. When the oblique axes originate at the centre, the equation is evidently

$$x^2+y^2+2xy\cos A = r^2 \quad . \quad . \quad . \quad (7)$$



In the more general equation (6), the origin is anywhere in the plane of the circle, and the angle A is of any magnitude, in consequence of which unrestriction we may readily deduce from (6) the properties of the circle which are demonstrated in Propositions 35, 36, and 37, of Euclid's third book; thus: let A be *within* the circle; then if in the equation (6) we put $x = 0$, and solve the resulting quadratic equation, the two roots or values of y will denote the two segments of the axis of y within the circle, these segments being one on one side of A and the other on the opposite side of A; and if in the same equation we put $y = 0$, the two resulting values of x will be the corresponding segments of the axis of x within the circle, and measured from the same point A also in opposite directions. Now, whether we put $x = 0$, or $y = 0$, in equation (6), the absolute term (that is, the term independent of x and y) in the one case is the same as the absolute term in the other, when r^2 is transposed to the left; this absolute term, thus common to both the equations, being obviously $\alpha^2+\beta^2+2\alpha\beta\cos A-r^2$. But the absolute term of any equation with a single unknown quantity, the coefficient of the highest power of that quantity being unit, as here, is the product of the roots of that equation; hence, the product (or rectangle) of the two segments of the one chord which passes through A, is equal to the product (or rectangle) of the two segments of the other chord passing through A; in other words, if two chords of a circle intersect, the rectangle of the segments of the one is equal to the rectangle of the segments of the other. Again, let the origin A be *without* the circle; then in a similar way is it shown that the product of the two segments of the axis of x, *between A and the* circumference, is equal to the product of the two segments of the axis of y between A and the circumference. When either axis *touches* the circle, the two corresponding roots are *equal*; and the rectangle of the other two segments is equal to the square of this touching line.

Problem 1.—*To determine the conditions which are to be fulfilled in order that two circumferences of circles may cut one another, or touch each other.*

Let O, O′, be the centres of the circles, and r, r', their radii; let also the distance O O′ between the centres be d.

Take the line through the centres for the axis of x, and a perpendicular to it O Y, from O, for the axis of y.

The circle, of which the radius is r, will have for its equation

$$y^2+x^2=r^2 \quad . \quad . \quad . \quad (1).$$

The other circle, of which the co-ordinates of the centre are d and 0, will have for its equation

$$y^2+(x-d)^2=r'^2 \quad . \quad . \quad . \quad (2).$$

The co-ordinates of the points in which these two circles intersect will be found by determining the values of x and y from (1) and (2), treated as simultaneous equations.

These values turn out to be

$$x=\frac{r^2-r'^2+d^2}{2d},$$

$$\begin{aligned} y &= \pm\frac{1}{2d}\sqrt{\{4d^2r^2-(r^2-r'^2+d^2)^2\}} \\ &= \pm\frac{1}{2d}\sqrt{\{(r^2+2dr+d^2-r'^2)[r'^2-(r-d)^2]\}} \\ &= \pm\frac{1}{2d}\sqrt{\{(r+d+r')(r+d-r')(r'+r-d)(r'-r+d)\}}. \end{aligned}$$

As the first factor under the radical sign is positive, y will be *real*, provided the three other factors are either all positive, or one of them positive, and the other two negative. But this last circumstance cannot happen, since if one of these three factors is negative the other two must be positive; thus, suppose the factor

$(r+d-r')<0$, then $(r+d)<r'$, $\therefore$ $r<r'$, and $d<r'$; consequently, $r'-r$ is positive, and $r'-d$ is also positive; much more are $r'-r+d$ positive, and $r'-d+r$, or $r'+r-d$ positive.

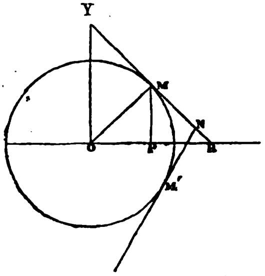
Hence 1. If one of the three factors be negative, since the other two must be positive, the expression above, under the radical sign, will be *negative*, and therefore the value of y, *imaginary*: there will be *no* point of intersection, so that the two circumferences must be separated by an interval.

2. If, however, there be no negative factor, the expression for y will be *real*, and the two values of it will be equal, but of opposite signs; so that the circumferences will intersect in two points, and the perpendicular to the line joining the centres, from one point, will pass through the other point, this perpendicular being the chord common to both circles.

3. If either of the factors be 0, the expression for y will be 0; so that the two circumferences will have only *one* point in common, and that is on the line joining the centres; the circumferences therefore *touch* at this point.

It follows, therefore, that when all the factors are positive, the circumferences intersect in *two* points; that is, when the sum of any and every two of the quantities r, r', d, is *greater* than the third; that the circumferences merely *touch* when one of the factors is *zero*, that is, when the sum of any two of these quantities is *equal* to the third; and that the circles are wholly separated, when the sum of any two of the above quantities is *less* than the third.

Equation of the tangent at a given point in the circumference of a circle.



Let the centre of the circle be taken for the origin of the rectangular co-ordinates, and let the point M, represented by (x', y'), be that at which the tangent is to be applied.

The equation of OM is $y = \frac{y'}{x'}x$, and since the tangent YB, passing through (x', y'), is perpendicular to this line, its equation is

$$y - y' = -\frac{x'}{y'}(x - x'), \text{ or } y'y + x'x = y'^2 + x'^2,$$

that is, $y'y + x'x = r^2$ is the equation of the tangent.

But, without availing ourselves of the geometrical property, that a tangent to a circle is perpendicular to the radius drawn to the point of contact, we may proceed otherwise as follows:

First suppose a secant to be drawn through M; that is, through (x', y'), and also through a second point (x'', y''); then (p. 13) the equation of this secant will be

$$y-y' = \frac{y'-y''}{x'-x''}(x-x') \quad . \quad . \quad . \quad (1),$$

and since both the points are on the circumference, they must satisfy the conditions

$$y'^2+x'^2 = r^2 \quad . \quad . \quad . \quad (2),$$
$$y''^2+x''^2 = r^2 \quad . \quad . \quad . \quad (3).$$

Subtract (3) from (2), and there results

$$y'^2-y''^2+x'^2-x''^2 = 0 \quad . \quad . \quad . \quad (4).$$

But since the difference of the squares of any two quantities is the product of the sum and difference of the quantities themselves, the equation (4) may be written

$$(y+y'')(y'-y'')+(x'+x'')(x'-x'') = 0,$$
$$\therefore \frac{y'-y''}{x'-x''} = -\frac{x'+x''}{y'+y''},$$

and in this way we get a more convenient expression for the co-efficient of $(x-x')$ than that given in equation (1); which equation may now be written

$$y-y' = -\frac{x'+x''}{y'+y''}(x-x') \quad . \quad . \quad . \quad (5),$$

a form which, when the two points (x', y'), (x'', y'') coalesce into a single point (x', y'), that is, when $x'' = x'$, and $y'' = y'$, becomes

$$y-y' = -\frac{x'}{y'}(x-x'), \text{ or } y'y+x'x = x'^2+y'^2$$
$$\text{or } y'y+x'x = r^2,$$

the equation of the tangent as before.

The object of changing the form of the co-efficient of $(x-x')$, in (1), for the form equivalent to it in (5), is that when $x'' = x'$, and $y'' = y'$, the indeterminate form $\frac{0}{0}$ may be replaced by a determinate form, namely, $-\frac{x'}{y'}$. Since the equation of OM, the line through the origin to the point (x', y'), is $y=\frac{y'}{x'}x$, we may now *infer* the property assumed in the former investigation, namely, that the line from the centre to the point of contact is perpendicular to the tangent.

Equation of the tangent to a circle from a given point without the circle.

Let (a, b) represent the given point P, when referred to rectangular axes, originating at the centre of the circle. It is required to find through what point (x', y') on the circumference, the line from P must pass in order that it may touch the circle.

The point (x', y') being on the circumference, we must have

$$x'^2+y'^2=r^2 \quad . \quad . \quad . \quad (1).$$

Also, since the point (a, b) is on the tangent at the point (x', y'), the equation of which tangent is

$$y'y+x'x=r^2, \therefore y'b+x'a, \text{ or } ax'+by'=r^2 \quad . \quad . \quad (2).$$

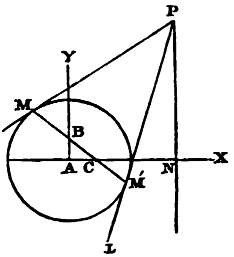
Now from the equations (1), (2), the co-ordinates x', y', of the point of contact M, may be determined; and the equation of the tangent PM, will obviously be that of the straight line passing through the two known points P and M. It is, however, evident, since the equation (1) is of the second degree, that there will be *two* values for x', and two corresponding values for y', so that there will be two points M and M′, to either of which if a straight line from P be drawn, the circle will be touched by that line; that is, from the same point P, *two* tangents to a circle may be drawn.

It is easy to see that the analytical representation of the co-ordinates x', y', would be complicated; but the two points of contact may be otherwise determined with but little trouble, and thence the two tangents be readily drawn; for all we have to do is to draw the line of which the equation is

$$ax+by=r^2 \quad . \quad . \quad . \quad (3); \text{ from which, for } y=0,$$

$$\text{we have } x=\frac{r^2}{a};$$

and for $x = 0$, $y = \frac{r^2}{b}$. And these values of x and y will be the intercepts AC, AB, of the axes, cut off by *the chord of contact*, MM′; for the equation (3) is that of a straight line necessarily passing through each of the two points M, M′, symbolised by the two-fold values of (x', y'); inasmuch as that x', being put for x, and y' for y, satisfy the equation (3), as we know these values do from equation (2) above. We thus see that



$$ax + by = r^2 \quad . \quad . \quad . \quad (3)$$

is the equation of the line MM′, *the chord of contact*; and to draw this chord, we have only to take on AX, $\text{AC} = \frac{r^2}{a}$; and on AY, $\text{AB} = \frac{r^2}{b}$; and then to draw the chord through the points B, C; this will meet the circumference in the points M, M′.

We have stated above that the algebraical expressions for the co-ordinates x' and y', would be complicated, as these expressions would result from actually solving the simultaneous equations (1), (2). The student may find for himself that they would have been

$$x' = \frac{ar^2}{a^2+b^2} \mp \frac{br}{a^2+b^2}\sqrt{(a^2+b^2-r^2)}$$

$$y' = \frac{br^2}{a^2+b^2} \pm \frac{ar}{a^2+b^2}\sqrt{(a^2+b^2-r^2)}.$$

Now the equation of a line passing through two points is of the form

$$y - y' = a'(x - x') \quad . \quad . \quad . \quad (4),$$

in which the co-efficient a' is equal to the difference of the ordinates of the two points divided by the difference of the abscissas of the same points: in the present case the difference of the ordinates is

$$+\frac{2ar}{a^2+b^2}\sqrt{(a^2+b^2-r^2)};$$

and the difference of the abscissas is

$$-\frac{2br}{a^2+b^2}\sqrt{(a^2+b^2-r^2)};$$

so that, dividing the first of these expressions by the second, we have for the value of the co-efficient a', $a' = -\frac{a}{b}$; and consequently the equation (4) is

$$y-y' = -\frac{a}{b}(x-x'), \text{ or } ax+by = ax'+by'.$$

But, equation (2), $ax'+by' = r^2$; therefore the equation of the line through the two points is

$$ax+by = r^2$$

as otherwise determined above. And the equation of the tangent (or rather the equation embodying the two tangents) is

$$y-y' = \frac{x'-a}{y'-b}(x-x'),$$

in which x', y', are the two expressions above.

A very interesting property of the circle is immediately deducible from the equality AC $= \frac{r^2}{a}$, which, being independent of the value of b, remains the same wherever in the perpendicular PL the point P be situated: the property is this, namely, if from each of any number of points in a straight line, pairs of tangents be drawn to a circle, the chords joining the points of contact of each pair will all intersect in the same point; this point being on the line through the centre of the circle, perpendicular to the line on which the tangents originate. It is a valuable peculiarity of Analytical Geometry that unsought-for properties frequently spontaneously offer themselves to our notice in this way.

In the foregoing diagram, the line PL is wholly *without* the circle; and the intersection of the several chords of contact is, in this case, always *within* the circle; because then AC, or $\frac{r^2}{a}$, is *less* than r; but if, on the contrary, PL cut the circle, then AC, or $\frac{r^2}{a}$, is *greater* than r; and the point of intersection is therefore *without* the circle.

THEOREM.—Every equation of the second degree of the form

$$x^2+y^2+\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0 \quad . \quad . \quad . \quad (1),$$

in which the product xy is absent, and x^2, y^2, have unit-coefficients, is the equation of a circle referred to rectangular axes.

Take the point whose co-ordinates are

$$a=-\frac{\mathrm{A}}{2}, \beta=-\frac{\mathrm{B}}{2},$$

for the centre of a circle with radius $r=\sqrt{\left\{\frac{\mathrm{A}^2+\mathrm{B}^2}{4}-\mathrm{C}\right\}}$: the circumference of this circle will be the locus of the equation (1).

For the equation of the circle thus constructed, is

$$(x-a)^2+(y-\beta)^2=r^2, \text{ that is, it is}$$

$$\left(x+\frac{\mathrm{A}}{2}\right)^2+\left(y+\frac{\mathrm{B}}{2}\right)^2=\frac{\mathrm{A}^2+\mathrm{B}^2}{4}-\mathrm{C} \quad . \quad . \quad . \quad (2)$$

and this, when developed, and the right-hand member transposed, is

$$x^2+y^2+\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0.$$

Hence the circle, constructed as above directed, is the locus of the equation (1); and thus any equation of the second degree, in which the squares of the variables have *equal coefficients*, and in which the *product* of the variables does not enter, may be affirmed to be the equation of a circle referred to rectangular axes of co-ordinates; and that such is not the case when the equation is not reducible to this form.

It should be observed, however, that when we say that every equation of the form (1) represents the circumference of a circle, we mean that this is the only kind of line it *can* represent. The coefficients may indeed be so related that the equation may admit of no geometrical interpretation at all; such is the case when C is greater than $\frac{\mathrm{A}^2+\mathrm{B}^2}{4}$; the equation then being

$$\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2 = a\ negative\ quantity,$$

which we know the sum of two squares cannot be; the radius r is then imaginary; there is no real circle. It is plain, however, that the locus cannot be imaginary, when C, the absolute term of the equation (1), is negative.

Again, if $\frac{A^2+B^2}{4}$, instead of being greater or less than C, be exactly equal to it, the radius r is 0; and the equation, which under this condition is

$$\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2 = 0,$$

represents merely a single point; the *centre*, into which the circle dwindles as $\frac{A^2+B^2}{4}$ diminishes in value down to the value of C. The equation being satisfied for $x = -\frac{A}{2}$, $y = -\frac{B}{2}$, we infer that these are the co-ordinates of the single point represented by it; the sum of two squares cannot be 0, unless each square is itself 0.

The general values of a, β, and r, inferred from the general equation (1) at page 47, may be used as formulas of reference in every particular case; but we would recommend the learner to dispense with these formulas, and to investigate each individual instance independently of them, as in the worked examples which follow.

Problems in reference to the Circle.

1. It is required to construct the locus of the equation

$$2x^2+2y^2-3x+4y-1 = 0.$$

As x^2, y^2, have equal coefficients, and xy is absent, we know that the equation is that of a circle. Dividing by 2, the equation becomes

$$x^2+y^2-\tfrac{3}{2}x+2y-\tfrac{1}{2} = 0,$$

so that the co-ordinates of the centre of the circle are $\frac{3}{4}$, and -1, and the radius of it is

$$r = \sqrt{\{(\tfrac{3}{4})^2+1^2+\tfrac{1}{2}\}} = \tfrac{1}{4}\sqrt{33} \text{ (page 47)};$$

if, therefore, with the point $(\frac{3}{4}, -1)$ as centre, and $\frac{1}{4}\sqrt{33}$ as radius, a circle be described, the circumference of it will be the locus of the proposed equation.

Or thus, without reference to the general formulas.

Completing the square, first as regards the terms involving x only, and then as regards those containing y only, in the equation

$$x^2-\tfrac{3}{2}x+y^2+2y=\tfrac{1}{2}$$

we have

$$x^2-\tfrac{3}{2}x+(\tfrac{3}{4})^2+y^2+2y+1=\tfrac{9}{16}+1+\tfrac{1}{2},$$

$$\text{or } (x-\tfrac{3}{4})^2+(y+1)^2=\tfrac{33}{16};$$

which is the equation of a circle, the radius of which is $\frac{1}{4}\sqrt{33}$, and the co-ordinates of the centre $\alpha=\frac{3}{4}$, $\beta=-1$.

2. Determine the locus of the equation

$$4x^2+4y^2-12x-8y+13=0,$$

$$\text{or } x^2+y^2-3x-2y+\tfrac{13}{4}=0.$$

Arranging the terms thus:

$$x^2-3x+y^2-2y=-\tfrac{13}{4};$$

and completing the squares, we have

$$x^2-3x+(\tfrac{3}{2})^2+y^2-2y+1=-\tfrac{13}{4}+\tfrac{9}{4}+1=0;$$

that is,

$$(x-\tfrac{3}{2})^2+(y-1)^2=0:$$

the proposed equation therefore represents a point only, namely, the point $(\frac{3}{2}, 1)$.

3. What is the geometrical interpretation of the equation

$$x^2+y^2+4x-2y+7=0?$$

$$\text{or, } x^2+4x+y^2-2y=-7.$$

Completing the squares,

$$x^2+4x+4+y^2-2y+1=-7+4+1=-2;$$

that is, $$(x+2)^2+(y-1)^2=-2.$$

Hence, the equation proposed has no geometrical meaning; for although it is a particular case of the general form of the equation to a circle, yet, as the radius of the particular circle is imaginary, namely, $\sqrt{-2}$, the circle has no real existence.

4. What is the equation of the straight line drawn from the point (1, 1) to touch the circle whose equation is $3x^2+3y^2=4$?

The general equation of the chord of contact is $ax+by=r^2$ (p. 45), where (a, b) is the point from which the two tangents to its extremities are drawn: hence, in the present case the two equations

$$3x^2+3y^2=4, \text{ or } x^2+y^2=\tfrac{4}{3}, \text{ and } x+y=\tfrac{4}{3}$$

are simultaneous for the unknown points of contact which are common both to the circle and to the two tangents. The values of x and y deduced from the equations in the usual way are

$$x=\frac{2+\surd 2}{3},\ y=\frac{2-\surd 2}{2}.$$

Giving the double sign to the radical, these are the co-ordinates of the two points of contact; and we have now to find the equations of the straight lines through (1, 1) and these points. From the general equation of a straight line through two points, we have, in the present case,

$$y-1=\frac{1-\dfrac{2-\surd 2}{3}}{1-\dfrac{2+\surd 2}{3}}(x-1),$$

$$\text{or } y-1=\frac{3-2+\surd 2}{3-2-\surd 2}(x-1)=\frac{1+\surd 2}{1-\surd 2}(x-1),$$

$$\text{that is, } y-1=(3\pm 2\surd 2)(1-x),$$

which equation represents the *two* tangents drawn from the given point to the given circle, the radius of this circle being $r=\frac{2}{\surd 3}$, or $\frac{2\surd 3}{3}$, and the co-ordinates of its centre 0, 0.

5. The base and vertical angle of a plane triangle being given, to find the locus of the intersections of straight lines from the vertices to bisect the opposite sides.

Let AB be the fixed base, and C one position of the opposite vertex (see the diagram to Prob. 2, p. 33); let CM be the line from C bisecting the base, and P a point in the required locus; then (p. 34) PM $=\frac{1}{3}$CM. For rectangular axes take AB, and a line from A perpendicular to AB, and let X, Y represent the co-ordinates of the point C in any position; and x, y those of P: put also a for the fixed base.

Now, by common geometry we know that the locus of C is the arc of that circle passing through the points A, B, which circumscribes the triangle ABC: hence if (α, β) denote the centre of that circle, the co-ordinates X, Y of C are thus related, namely,

$$\mathrm{X}^2+\mathrm{Y}^2-2(\alpha\mathrm{X}+\beta\mathrm{Y})=0.$$

If to AB, a perpendicular Pp, be drawn from P, and a perpendicular Cc, from C, we shall have

$$\mathrm{M}c=3\mathrm{M}p\text{; that is, }\tfrac{1}{2}a-\mathrm{X}=3(\tfrac{1}{2}a-x),\ \therefore\ \mathrm{X}=3x-a.$$

Also because CM $=3$PM, Y $=3y$; therefore substituting these values in the foregoing equation, we have

$$(3x-a)^2+(3y)^2-2(3\alpha x-\alpha a+3\beta y)=0,$$

that is,

$$9x^2+9y^2-6(a+\alpha)x-6\beta y+a(a+2\alpha)=0,\quad . \quad . \quad . \ (1)$$

the equation of the *circle* which is the locus sought. Since the perpendicular from the centre (α, β) of the circle circumscribing the triangle ABC, bisects the base AB, it is plain that $\frac{a}{2}$, and α, are one and the same thing; namely, half the base: the equation of the locus may therefore be written

$$x^2+y^2-ax-\tfrac{2}{3}\beta y+\tfrac{2}{9}a^2=0\quad . \quad . \quad . \ (2).$$

The centre of this circle is the point $(\frac{1}{2}a, \frac{1}{3}\beta)$. To find where the arc meets the base AB, put $y=0$, in the equation of it; and we have

$$x^2-ax+\frac{2}{9}a^2=0;$$

$$\therefore\ x^2-ax+\frac{a^2}{4}=\frac{a^2}{4}-\frac{2a^2}{9}=\frac{a^2}{36},$$

$$\therefore\ x-\frac{a}{2}=\pm\frac{a}{6},\ \therefore\ x=\frac{1}{3}a,\text{ or }\frac{2}{3}a.$$

So that if the base AB be divided into three equal parts, the middle one of these parts will be the chord of the arc which is the locus of P; and the distance of each extremity of this chord, from its middle point M, will be $\frac{1}{3}$ MA, or MB; and since PM is $\frac{1}{3}$ MC, it follows that lines from P, to the ends of chord, will be parallel to the two sides of the triangle, each to each; therefore, the angle at P will always be equal to the given angle C; therefore, to construct the locus, we have only to trisect the base, and to describe upon the middle portion an arc to contain the given angle C. We may further notice that the distance of the centre of the locus from the base AB, is equal to one-third the distance of the centre of the circumscribing circle from AB.

The student will have observed that here, as well as in many other geometrical problems treated analytically, the result of the investigation has an interpretation beyond the geometrical restrictions of the problem proposed for solution: this arises from the very general character of algebraical symbols. The foregoing analysis would have been just as it is if the problem had been:—"Given the chord AB of a circle, and from the middle point M of this chord let lines to the circumference be drawn; and in each line let P, at the distance of one-third of its length from M, be taken: required the locus of P?" The locus would be the entire circumference (1) or (2) above: in the problem, as proposed, it is only an *arc* of this circumference that, from the geometrical restrictions, is the locus of P.

In every such enquiry the analytical result of our investigation will always supply the information we are in quest of, and frequently much more.

6. Find the centre and radius of the circle whose equation, referred to rectangular co-ordinates, is $x^2+y^2-8x+2y-8=0$? *Ans.* Centre, $(4, -1)$; radius, 5.

7. Required the centre and radius of the circle whose equation referred to rectangular co-ordinates is

$$2x^2+2y^2+12x-2y-6=0?$$

Ans. Centre, $(-3, \frac{1}{2})$; radius, $3\frac{1}{2}$.

8. Required the centre and radius of the circle whose equation is $x^2+y^2-ax-\frac{2}{3}by+\frac{2}{9}a^2=0$?

Ans. Centre, $(\frac{1}{2}a, \frac{1}{3}b)$; radius, $\frac{1}{6}\sqrt{(a^2+4b^2)}$.

9. Required the centre and radius of the circle whose equation is $(m-1)y^2+(m-1)x^2-2amx+ma^2=0$?

Ans. Centre, $\left(\frac{am}{m-1}, 0\right)$; radius, $\frac{a\sqrt{m}}{m-1}$.

10. From a given point in the circumference of a circle, chords are drawn: required the locus of their middle points?
Ans. A circle of half the diameter.

11. Given the base and the sum of the squares of the other two sides of a triangle to determine the locus of the vertex.

Ans. A circle of which the centre is at the middle of the base, and of which the square of the radius is half the sum of the squares of the sides minus the square of half the base.

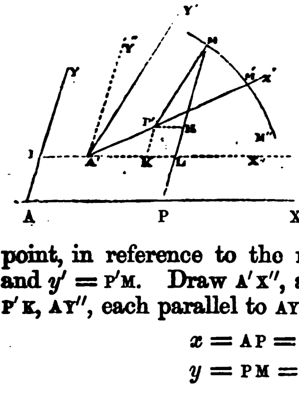
12. The base and vertical angle of a triangle being constant, prove that the locus of the intersection of the perpendiculars from the vertices to the opposite sides is a circle.

13. Prove that the loci of the four vertices of parallelograms of constant area, circumscribing a circle, are also circles.

SECTION VI.—ON THE TRANSFORMATION OF CO-ORDINATES.

It has been sufficiently seen in the foregoing sections, that the equation of a straight line, as also the equation of a circle, each assumes a simpler form when the axes of reference are rectangular than when they are oblique; and further, that to attain the utmost simplicity of form, the origin of the rectangular axes must not be fixed upon at random. In the case of the straight line, it must be a point upon the line itself; and in the case of the circle, it must be at the centre.

It is easy to foresee that in other curves simplicity of analytical representation may in like manner much depend upon a judicious choice of axes and of the point in which they originate; and that in general rectangular axes are to be preferred. Yet for the deduction of certain properties oblique axes of reference are better suited than rectangular axes; we have seen an exemplification of this at page 40, where, by employing oblique axes, properties of the circle which would not have offered themselves to notice if the axes had been rectangular, are very readily inferred from the equation. A change of axes from rectangular to oblique and *vice versâ*, or from one system to another system of the same kind, will often thus unfold to us new and sometimes unexpected properties of a curve, and hence the advantage when a curve is represented by an equation, of knowing how to transform that equation into another equally representing the curve, but in reference to whatever new axes we may choose to select. Such a transformation is called *the transformation of co-ordinates.* It may consist either in altering the inclination of the axes without displacing the origin, or in removing the origin without disturbing the relative position of the axes; or lastly, in changing both the inclination of the axes and the situation of the origin. We shall now investigate the formulæ to be employed for bringing about each of these several changes in the equation of any curve.



Let AX, AY be the axes to which any line MM'M'' is related by its equation; and let A'X', A'Y', be the new axes to which it is proposed to refer the same line. Let the co-ordinates of any point, M, in the line, relative to the primitive axes AX, AY, be x and y; and the co-ordinates of the same point, in reference to the new axes A'X', A'Y', be $x' =$ A'P', and $y' =$ P'M. Draw A'X'', and PH, each parallel to AX; and P'K, AY'', each parallel to AY. Then we shall have

$$x = \mathrm{AP} = \mathrm{BA}' + \mathrm{A}'\mathrm{K} + \mathrm{P}'\mathrm{H}$$
$$y = \mathrm{PM} = \mathrm{AB} + \mathrm{KP}' + \mathrm{HM},$$

in which equations BA' and AB are known, these being the co-ordinates a and b, of the origin A', of the new axes, when A' is referred to the primitive axes. The following quantities also are understood to be known, namely, the inclination of the primitive axes, and the inclination of each of the new axes to the primitive axis of x. It remains, therefore, to express A'K, KP', P'H, and HM, in terms of these known quantities.

In order to this, put

$\alpha =$ X'A'X'', the inclination of the new axis of x to the primitive axis of x,

$a' = \mathrm{Y'A'X''}$, the inclination of the new axis of y to the primitive axis of x. $\therefore\ a' - a = \mathrm{Y'A'X'}$,

$\beta = \mathrm{Y''A'X''}$ the inclination of the primitive axes to each other,

$\therefore\ \beta - a = \mathrm{Y''A'X'}$, the inclination of the new axis of x to the primitive axis of y,

$\beta - a' = \mathrm{Y''A'Y'}$, the inclination of the new axis of y to primitive axis of y.

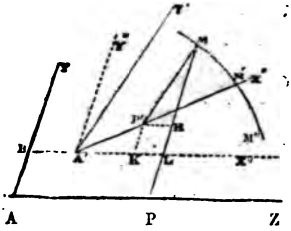
Now (*Trigonometry*, p. 52), the value of $\mathrm{A'K}$, in the triangle $\mathrm{A'P'K}$, is

$$\mathrm{A'K} = \frac{\mathrm{A'P'} \sin \mathrm{P'A'Y''}}{\sin \mathrm{A'KP'}} = \frac{x' \sin(\beta - a)}{\sin \beta}.$$

And the value of $\mathrm{KP'}$ in the same triangle is

$$\mathrm{KP'} = \frac{\mathrm{A'P'} \sin \mathrm{P'A'X}}{\sin \mathrm{A'KP'}} = \frac{x' \sin a}{\sin \beta},$$

observing that $\mathrm{A'P'K} = \mathrm{Y''A'X'} = \beta - a$; and that $\mathrm{Y''A'X''} = \beta$, is the supplement of $\mathrm{A'KP'}$.



In like manner, in the triangle $\mathrm{MP'H}$, we have for $\mathrm{P'H}$,

$$\mathrm{P'H} = \frac{\mathrm{P'M} \sin \mathrm{P'MH}}{\sin \mathrm{MHP'}} = \frac{y' \sin(\beta - a')}{\sin \beta},$$

and for HM,

$$\mathrm{HM} = \frac{\mathrm{P'M} \sin \mathrm{MP'H}}{\sin \mathrm{MHP'}} = \frac{y \sin a'}{\sin \beta}.$$

Hence for the values of x and y, we have

$$x = a + \frac{x' \sin(\beta - a) + y' \sin(\beta - a')}{\sin \beta}$$

$$y = b + \frac{x' \sin a + y \sin a'}{\sin \beta}.$$

These therefore are the values which must be substituted for x and y, in the equation of the curve, when referred to the primitive axes AX, AY, in order that that equation may be transformed into another which shall be the equation of the same curve when it is referred to the new axes $\mathrm{A'X'}$, $\mathrm{A'Y'}$, whatever be the inclinations of the two pairs of axes. The following are the most useful modifications of these general expressions.

1. *When the new axes are parallel to the primitive axes.*

In this case, since the inclination of the axes remains unaltered, while the origin is removed, we have $\alpha=0$, and $\beta-\alpha=\beta$, so that the general expressions for x and y become

$$\left.\begin{aligned} x &= a+x \\ y &= b+y \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

where a, b, are the x, y, of the new origin: and x, y, the new co-ordinates of any point (x, y) of the primitive equation.

2. *When the primitive axes are rectangular, and the new axes oblique.*

In this case $\sin(\beta-\alpha)=\cos\alpha$: and $\sin(\beta-\alpha')=\cos\alpha'$; therefore

$$\left.\begin{aligned} x &= a+x\cos\alpha+y\cos\alpha' \\ y &= b+x\sin\alpha+y\sin\alpha' \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

3. *When both systems are rectangular.*

In this case $\alpha'=90^\circ+\alpha$ $\therefore$ $\sin\alpha'=\cos\alpha$.

Also $\sin(\beta-\alpha')=\sin(-\alpha)=-\sin\alpha$: and $\sin(\beta-\alpha)=\cos\alpha$. Hence in this case the expressions become

$$\left.\begin{aligned} x &= a+x\cos\alpha-y\sin\alpha \\ y &= b+x\sin\alpha+y\cos\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

4. *When the primitive axes are oblique, and the new axes rectangular.*

In this case $\alpha'=90^\circ+\alpha$ $\therefore$ $\sin\alpha'=\cos\alpha$: also, since the complement of $(\beta-\alpha')$ is $90^\circ-(\beta-\alpha')=180^\circ-(\beta-\alpha)$, we have $\sin(\beta-\alpha')=-\cos(\beta-\alpha)$: hence the formulas for this case are

$$\left.\begin{aligned} x &= a+\frac{x\sin(\beta-\alpha)-y\cos(\beta-\alpha)}{\sin\beta} \\ y &= b+\frac{x\sin\alpha+y\cos\alpha}{\sin\beta}. \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

It will of course be borne in mind that, in each of these cases, if the origin remain fixed, the values of a, b, are each zero, and that A′X′ is always *above* A′X″, or α *positive*.

In applying any of the above formulas to the equation of any particular curve, the special interpretations of the angular quantities, here represented generally by α, α', and β, will always be readily ascertained by merely recollecting the conventions, as to these symbols, at page 55; that is, by recollecting that β is the inclination of the primitive axes to each other; that α is the inclination of the new axis of x to the primitive axis of x; and α' the inclination of the new axis of y to the primitive axis of x. When A′X′ is *below* A′X″, the angle α is *negative*.

Suppose we wished to transform the equation $x^2+y^2=r^2$, of a circle, referred to rectangular axes originating at the centre, into another equation, referring the circle to oblique axes having their origin at the same point. The formulas to be used for this transformation are those marked (2) above, when the a and the b are expunged; because here, there being no change of origin, these quantities are each zero; we have, therefore, to replace the x and y in the above primitive equation by

$$x\cos\alpha+y\cos\alpha',$$
$$\text{and } x\sin\alpha+y\sin\alpha',$$

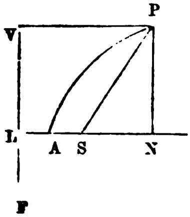
respectively; the dash over the x, and over the y, need not here be retained. By making these substitutions, the transformed equation is

$$\left.\begin{array}{l} x^2\cos^2\alpha+2xy\cos\alpha\cos\alpha'+y^2\cos^2\alpha'+\\ \quad x^2\sin^2\alpha+2xy\sin\alpha\sin\alpha'+y^2\sin^2\alpha'\end{array}\right\}=r^2$$

that is (*Trigonometry*, p. 25), $x^2+y^2+2xy\cos(\alpha'-\alpha)=r^2$ $=x^2+y^2+2xy\cos A'$, as at page 40. We shall leave the student to effect for himself the more general transformation of the above primitive equation when the co-ordinates of the new origin, in reference to the primitive axes, are a, b; he will of course find the result of the substitutions to agree with the form (6), at page 40, when the co-ordinates of the centre there, which are denoted by α, β, are exchanged for a, b, here.

SECTION VII.—THE PARABOLA.

If there be taken a fixed line VF, and a fixed point S, so that $SP = PV$, the perpendicular distance of P from the line VF, the locus of P, is the *parabola*. Through S draw the perpendicular SL, and bisect it in A, then it is evident that A is a point in the curve.



To find the equation to the parabola. Referring to rectangular co-ordinates, let A be the origin, $AN = x$, $NP = y$, $AL = AS = a$;

$$\text{then } VP = LN = SP,$$

$$\therefore LN^2 = (x+a)^2 = SP^2 = y^2 + (x-a)^2, \therefore y^2 = 4ax,$$

which is the equation of the parabola in reference to the assumed axes.

If S be the origin, then it is evident that $y^2 = 4a(x+a)$, for in that case $SN = x$.

If L be the origin, then $y^2 = 4a(x-a)$, since LN is then $= x$.

Polar Equations.

There is also what is called *the polar equation*, in which the variables are a line called a *radius vector* drawn from some fixed point, as S, to a point P in the curve, and the angle at S between this line and the axis SN.

If S, called the *focus*, be the pole, and $PSN = \theta$,

$$\text{then } r = SP = LN = 2a + r\cos\theta, \therefore r = \frac{2a}{1-\cos\theta},$$

$$\text{or } r = \frac{a}{\sin^2\frac{\theta}{2}} \text{ (Hann's \textit{Trigonometry}, p. 27),}$$

which is the polar equation, S being the pole.

It is evident that since $4a$ may be made to equal any constant quantity, the equation to the parabola may be written $y^2 = px$. If $x = a$, then the corresponding value of y is $2a$, therefore the double ordinate through the focus is $p = 4a$. This double ordinate is generally termed the *parameter*, or *latus rectum*.

PROPERTIES OF THE PARABOLA.

Properties deduced from the Tangent.

The equation of a straight line passing through (x', y'), (x'', y''), is

$$y-y' = \frac{y'-y''}{x'-x''}(x-x') \quad . \quad . \quad . \quad (1),$$

and by the equation of the parabola, if these points are on the curve,

$$y'^2 = 4ax' \quad . \quad . \quad . \quad (2),$$
$$\text{and } y''^2 = 4ax'' \quad . \quad . \quad . \quad (3);$$

therefore, subtracting (3) from (2),

$$(y'+y'')(y'-y'') = 4a(x'-x''),$$
$$\therefore \frac{y'-y''}{x'-x''} = \frac{4a}{y'+y''},$$

hence, by substituting, (1) becomes

$$y-y' = \frac{4a}{y'+y''}(x-x').$$

Now, let the points (x', y'), (x'', y''), approach indefinitely near to one another, then the secant joining these two points becomes a tangent, and its equation is

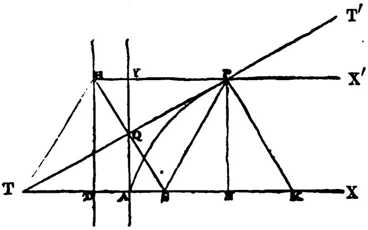
$$y-y' = \frac{2a}{y'}(x-x') \quad . \quad . \quad . \quad (4).$$

This equation may be put under a different form; for multiplying by y', and transposing,

$$\begin{aligned} yy' &= 2ax-2ax'+y'^2 \\ &= 2ax-2ax'+4ax' \\ &= 2a(x+x') \quad . \quad . \quad . \quad (5), \end{aligned}$$

an equation closely analogous to the equation of the curve, and only differing from it in this, that y^2 and $2x$ are replaced by yy' and $(x+x')$.

In the annexed figure, suppose P to be a point in the curve, AN $= x$, NP $= y$, and PT a tangent to the curve at P, cutting the axis of x at T. Then NT is called the *subtangent* to the point P, and if PK, perpendicular to PT, be drawn cutting the axis of x at K, PK is termed the *normal*, and NK the *subnormal*. The fixed line HD is called the *directrix*.



Now, from (5), when $y = 0$, $x = -x'$, therefore $\text{NT} = 2x'$, therefore, the subtangent is equal to twice the abscissa; a very useful and important property.

Since the normal is at right angles to the tangent, it follows from (4), that $y - y' = \frac{-y'}{2a}(x - x')$ is the equation of the normal.

To find an expression for the length of the subnormal NK, put $y = 0$ in its equation, $\therefore\ x - x' = 2a = \text{NK}$, a *constant.*

The length of the tangent is $\sqrt{\text{NP}^2 + \text{NT}^2} = 2\sqrt{ax' + x'^2}$

„ „ „ normal is $\sqrt{\text{NP}^2 + \text{NK}^2} = 2\sqrt{ax' + a^2}.$

We now proceed to the investigation of some properties of the parabola.

When $x' = a$, that is, when the tangent passes through the extremity of the parameter, the tangent and normal are equal, and also the subtangent and subnormal,

$$\text{the equation of the tangent is then } y - 2a = (x - a),$$

$$\therefore\ y = (x + a).$$

Therefore, any ordinate to the *focal* tangent, as this is called, = rad. vector of the point where the ordinate cuts the curve, so that the focal tangent cuts off from the tangent through the vertex A, a part equal to the distance of the vertex from the focus S.

Draw the line PH perpendicular to the directrix HD.

Then $\text{SP} = \text{HP} = \text{DN} = \text{TS}$, and HTSP is a parallelogram with equal sides, and for the angle HPT we have

$$\cos \mathrm{HPT} = \frac{\mathrm{HP}^2 + \mathrm{PT}^2 - \mathrm{ST}^2}{2\mathrm{HP} \,.\, \mathrm{PT}} = \frac{\mathrm{PT}^2}{2 \,.\, \mathrm{HP} \,.\, \mathrm{PT}} = \frac{\mathrm{PT}^2}{2\mathrm{SP} \,.\, \mathrm{PT}},$$

$$= \frac{\mathrm{PT}}{2\mathrm{SP}}.$$

The triangle HPS being isosceles, PT bisects the angle HPS, and it also bisects HS at right angles in Q; the point Q is therefore always on the axis of ordinates AY;

$$\therefore \angle \mathrm{HPT} = \angle \mathrm{SPT},$$

therefore, the tangent at any point bisects the angle made by two straight lines, one drawn to the focus, and the other perpendicular to the directrix. Again,

$$\mathrm{ST} = \mathrm{AS} + \mathrm{AT} = a + x',$$

$$\mathrm{SK} = \mathrm{SN} + \mathrm{NK} = x' - a + 2a = x' + a,$$

$$\therefore \mathrm{ST} = \mathrm{SK}, \text{ but } \mathrm{ST} = \mathrm{SP},$$

$$\therefore \mathrm{SP} = \mathrm{ST} = \mathrm{SK}.$$

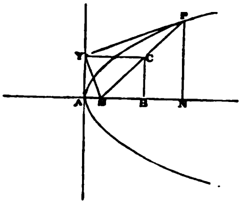
From P draw PX′ parallel to the axis AX, then

$$\mathrm{T'PX'} = \mathrm{PTS} = \mathrm{SPT},$$

$$\therefore \mathrm{KPX'} = \mathrm{SPK}.$$

Therefore the radius vector and the diameter PX′ at the point of contact are equally inclined to the tangent.

These are some of the principal parabolic properties; others will be furnished by the following theorems.



Theorem 1.—The focus S and any point P being joined, and a circle described on SP as diameter, this circle shall touch the axis AY.

Through P draw a tangent cutting the axis of (y) in Y; through Y draw YC perpendicular to AY, then from the equation to the tangent

$$yy' = 2a(x + x')$$

and the equation to AY, being $x = 0$,

$$\text{we find } \mathrm{AY} = y = \frac{2ax'}{y'} = \frac{y'}{2}$$

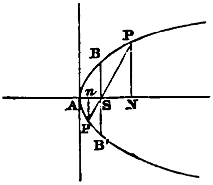
(since $x'y'$ is a point in the parabola),

$$\therefore \mathrm{BC} = \frac{\mathrm{PN}}{2}, \quad \therefore \mathrm{SC} = \mathrm{CP},$$

or C is the centre of the circle, and $CY = CP$; for the angle $CPY =$ the angle CYP, for it is $=$ the angle at T (last fig.); but CY is perpendicular to AY, therefore the circle touches AY.

THEOREM 2.—Any chord Pp being drawn through S, the focus to a parabola, show that

$$4SP \,.\, Sp = L(SP+Sp),$$



L being the latus rectum, or double ordinate (BB') through the focus,

$$SP = AN+AS,$$
$$Sp = An+AS;$$

and (by sim. tri.) $SP : Sp :: SN : Sn$
$$:: (AN-AS) : (AS-An)$$
$$:: (SP-2AS) : (2AS-Sp),$$
$$\therefore\; SP(2AS-Sp) = Sp(SP-2AS),$$
$$\therefore\; 2SP \,.\, Sp = 2AS(SP+Sp).$$

But $4AS = BB' = L$,
$$\therefore\; 4SP \,.\, Sp = L\,(SP+Sp).$$

Properties of the Parabola, when related to conjugated axes.

The equation of the parabola, $y^2=px$, the axes being AX, AY, originating at the vertex A (see diagram at p. 60), shows that the curve is of unlimited extent towards the right, and that the axis of x, or the principal diameter as it is called, bisects all the chords drawn parallel to the axis of y; for whatever positive value we give to x, there will always correspond *two* real values for y, namely, $y = \sqrt{px}$, and $y = -\sqrt{px}$, which are equal in length but of opposite signs. But towards the left of the axis of y there does not exist any point belonging to the curve, since a *negative* value of x gives, for the corresponding y, an impossible quantity, namely, $y = \sqrt{-px}$.

A straight line from a point in a curve which bisects a series of parallel chords is called a *diameter*, though, as in the case of the parabola, this straight line never again meets the curve: we shall presently see that the diameters of a parabola, like those of a circle, are innumerable; in fact, that whatever line be drawn parallel to AX, there is always an infinite number of parallel chords, all of which will be bisected by that line: we now proceed to show this to be the case.

Let the equation $y^2 = px$ be transformed from rectangular to oblique axes by means of the formulas (2) p. 56, namely,

$$x = a + x' \cos \alpha + y' \cos \alpha'$$
$$y = b + x' \sin \alpha + y' \sin \alpha'.$$

The transformed equation, suppressing the dashes, will be

$$\left.\begin{array}{l} y^2 \sin^2\alpha' + 2xy \sin \alpha \sin \alpha' + x^2 \sin^2\alpha + b^2 - ap \\ + (2b \sin \alpha' - p \cos \alpha')y + (2b \sin \alpha - p \cos \alpha)x \end{array}\right\} = 0 \ldots (1)$$

and in order that it may be converted into the same form as the primitive equation, that is, into the form $y^2 = p'x$, we have only to subject the four disposable constants, α, α', a, and b, to the four conditions

$$\sin \alpha \sin \alpha' = 0,\ \sin^2\alpha = 0,\ 2b \sin \alpha' - p \cos \alpha' = 0,$$
$$b^2 - ap = 0 \quad . \quad . \quad . \ (2)$$

since, under these conditions, the terms involving xy, x^2, and y, respectively will disappear, as also the term $(b^2 - ap)$, which is independent of the variables, and the equation (1) will be reduced simply to

$$y^2 \sin^2\alpha' - px = 0; \text{ since } \cos \alpha = 1;$$
$$\text{inasmuch as that } \sin \alpha = 0.$$

Hence, $y^2 = \dfrac{p}{\sin^2 \alpha'}\, x$, which is of the form $y^2 = p'x$.

The conditions (2) furnish ample information as to the directions of the axes of reference, and the situation of their origin, in order that the equation of the parabola may have this simple form:

1. From the second of the above conditions (which second condition necessarily establishes the first) we see that the new axis of x must be parallel to the primitive axis of x; for the sine of the inclination of the one to the

other is 0; therefore α' represents the inclination of the new axes themselves.

2. This inclination α' is determined from the third condition; for to fulfil this, we must have

$$\frac{\sin\alpha'}{\cos\alpha'}, \text{ that is } \tan\alpha' = \frac{p}{2b} \quad . \quad . \quad . \ (3).$$

3. From the fourth condition we learn the relation which must of necessity exist between the co-ordinates of the new origin, in order that the primitive form of the equation of the curve may be preserved: that fourth condition shows that the relation of the co-ordinates a, b, must be

$$b^2 = pa \quad . \quad . \quad . \ (4),$$

from which we infer that the origin of the new axes must be upon the curve; but it is not restricted to any *particular* point, as a may be any positive value whatever; that is to say, the point (a, b) may be any point whatever, provided it be not out of the curve: hence, from the relation (3), which involves the value of b, the inclination α' of the new axes may take an indefinite number of values; so that the systems of co-ordinates that will render the equation of the parabola of the form $y^2 = p'x$, are innumerable; but every system must have its origin on the curve; and its axis of x must be parallel to the principal diameter AX of the curve. This axis of x is itself a diameter; since the form of the equation shows that for every value of x there are *two* values of y, equal in length but opposite in sign; that is, the axis of x bisects all the chords drawn parallel to the axis of y: there are thus innumerable diameters to a parabola, and all of them are parallel to the principal diameter.

Since, from the form of the equation, y is 0 whenever x is 0, it follows that in every system of co-ordinate axes, the axis of y is a tangent to a curve at the origin. The two axes of any such system are called *conjugate axes*.

$$\text{Since from (3) } \tan^2\alpha' = \frac{p^2}{4b^2},$$

$$\therefore \sec^2\alpha' = 1 + \frac{p^2}{4b^2} = \frac{4b^2+p^2}{4b^2},$$

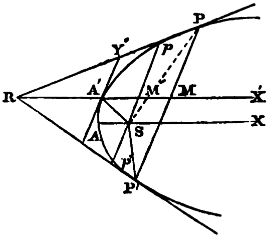
$$\therefore \frac{\sec^2 a'}{\tan^2 a'} = \frac{1}{\sin^2 a'} = \frac{4b^2+p^2}{p^2} = \frac{4a+p}{p} \text{ (Equation 4),}$$

$$\therefore \frac{p}{\sin^2 a'} = 4a+p = 4\left(a+\frac{p}{4}\right).$$

Now (p. 58) $a+\frac{1}{4}p$ expresses the distance of a point A′ in the curve, of which the abscissa is a, from the focus, that is, the radius vector of the point A′; therefore representing this radius vector by r, the equation $y^2 = \frac{p}{\sin^2 a'}$, of the curve (p. 63) may be written

$$y^2 = 4rx \quad . \quad . \quad . \quad (5),$$

which is the general equation of the parabola when related to any system of conjugate axes, in terms of the radius vector of the origin of those axes. It of course comprehends the equation at p. 58, where the axes are the *principal conjugates*, the radius vector of their origin being there represented by a: the principal or *right parameter* p, was there shown to be $p = 4a$; similarly, the coefficient $4r$, whatever system of conjugate axes be employed, is called the *parameter of that diameter* which is taken for the axis of x; so that, generally, the parameter of any diameter is always equal to four times the distance of the vertex of that diameter from the focus. Moreover, from the general equation (5), we see that the parameter is always equal to the double ordinate corresponding to the abscissa $x = r$; and we shall now show that, as in the case of the axes being the *principal* axes, this double ordinate or parameter passes through the focus; so that, generally, whatever be the conjugate axes to which the curve is referred, the parameter of the diameter taken for the axis of x always passes through the focus.



Let A′M′ $= r$, and the parabola be referred to its principal axes originating in the point A: then the abscissa of A′, the origin of the conjugate axes, A′X′, A′Y′, being a, we have, for the abscissa of M′, the expression $a+r$; but r, the radius vector of the point A′, is $a+m$ (m being put for AS); hence the abscissa of M′ is $2a+m$, and its ordinate is b. Also for the tangent of a', the inclination of the chord pp', parallel to A′Y′, we have (p. 63),

$$\tan a' = \frac{p}{2b};$$

therefore the equation of pp' is $y-b = \frac{p}{2b}(x-2a-m)$.

Now at the point where pp' intersects AX, we must have $y=0$; for which value the equation gives

$$x = 2a+m-\frac{2b^2}{p} = 2a+m-2a = m = \text{AS}.$$

Hence the chord through the focus is always the parameter of the diameter which bisects that chord.

We shall now recapitulate the foregoing properties of the parabola, introducing a few other properties for the student himself to investigate.

Summary of the Properties of the Parabola.

1. Abscissas measured from the vertex of any diameter, are as the squares of the corresponding ordinates.
2. At whatever point in the curve a tangent be drawn, the subtangent is always equal to twice the abscissa of that point.
3. Two tangents from P, P′, the extremities of any chord, meet the diameter bisecting that chord in the same point R (see diagram, p. 65).
4. The tangent at A′, intercepted between the two tangents from P, P′, is bisected at the point of contact A′.
5. For every point in the curve the subnormal is constant, and equal to the distance of the focus from the directrix.
6. The subnormal, in reference to any point of a parabola, is always of the same length, namely, half the principal parameter.
7. Chords drawn parallel to a tangent are all bisected by the diameter from the point of contact.
8. The diameter through the point where two tangents meet bisects the chord of contact.

9. If the chord of contact of two tangents pass through the focus, the tangents will meet in a right angle, the vertex of which will be on the directrix.

10. That part of a tangent which is intercepted between the point of contact and the directrix subtends a right angle at the focus.

11. A perpendicular from the foot of the normal, to the radius vector, cuts from that radius a part, towards the curve, equal to the semiparameter.

12. The locus of the intersections of perpendiculars from the focus on tangents to a parabola, is the line which touches the curve at its vertex.

This property we shall prove, thus: Let $y^2 = 4ax$ be the equation of the parabola; then the point $(a, 0)$ will be the focus; and we shall have

Equation of tangent at (x', y')

$$y'y = 2ax + 2ax'$$

$$\therefore y = \frac{2a}{y'}x + \frac{2ax'}{y'} \quad . \quad . \quad . \quad (1),$$

Equation of perp. to (1) from $(a, 0)$

$$y = -\frac{y'}{2a}(x-a) \quad . \quad . \quad . \quad (2),$$

Subtract (2) from (1), and there results

$$0 = \left(\frac{2a}{y'} + \frac{y'}{2a}\right)x + \frac{2ax'}{y'} - \frac{y'}{2} \quad . \quad . \quad . \quad (3).$$

But, since by the equation of the curve,

$$4ax' - y'^2 = 0, \therefore \text{ (dividing by } 2y'),$$

$$\frac{2ax'}{y'} - \frac{y'}{2} = 0 : \text{ hence (3)},$$

$$\left(\frac{2a}{y'} + \frac{y'}{2a}\right)x = 0, \therefore x = 0.$$

Consequently, the intersection is always on the axis of y; that is, on the tangent to the parabola at its vertex.

We may proceed a little differently, as follows:

The abscissa of the focus being represented by a, and m being put for an arbitrary constant, the general equation of a straight line through the focus may be written

$x = my + a$; since for $y = 0$, in this equation, $x = a$; therefore $y = \frac{1}{m}(x-a)$, the coefficient $\frac{1}{m}$ being arbitrary, unless the line through the focus is to be subject to some condition. In the present case, the line is to fulfil the condition of perpendicularity to the line (1) above; the value of the coefficient must therefore be $-\frac{y'}{2a}$: hence the equation of the perpendicular to (1), though the focus, is

$$y = -\frac{y'}{2a}(x-a)$$

as above: this equation being obtained, the remaining steps are the same as before. Of course the coefficient of inclination here is the same as that in the equation of the normal at the point (x', y').

For the value $x = 0$, $y = \frac{y'}{2}$; so that the ordinate of the point of contact is always equal to twice the intercept of the axis of y, between the origin and this perpendicular; it is also equal to twice the intercept cut off by the tangent itself; for putting $x = 0$, in the equation of the tangent we have $y = \frac{2ax'}{y'} = \frac{\frac{1}{2}y'^2}{y'} = \frac{1}{2}y'$; so that the tangent and the perpendicular to it from the focus both cross the axis of y at the same point; and in this way the locus of the intersections might have been shown to be the tangent at the vertex of the curve.

13. If from the focus perpendiculars to focal chords are drawn, the locus of the intersections of these perpendiculars with the corresponding tangents at the extremities of the chords, will be the directrix of the parabola.

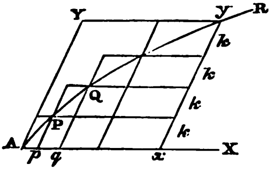
14. Any rectangular ordinate to the focal tangent is equal to the radius vector of the point where that ordinate cuts the curve.

15. If from any point two secants, each cutting the parabola in two points, be drawn parallel to the sides of a fixed angle, the rectangle of the parts, between the point and the curve, of one secant, will be in a constant ratio to the *rectangle of* the like parts of the other secant, wherever the

point from which the secants are drawn be situated. [This is true for each of the conic sections.]

Area of the Parabola.

Let the parabola APQR be referred to the conjugated axes AX, AY; the axis AX, being a diameter of the curve, and AY a tangent at the extremity, A, of that diameter; and let it be required to find the area of the space Axy, included between the arc Ay, and the co-ordinates Ax, xy, of its extremity, y; that is, the area of the parabolic surface Axy.



Divide the line xy into any number n of equal parts k, k, k, &c.; and from the extremities of these parts, let parallels to Ax be drawn, to terminate in the curve, as in the diagram at the points P, Q, &c.; and let the ordinates Pp, Qq, &c., of these several points be drawn. Then it is obvious that Pp, parallel to yx, will be equal to one of these parts (k); and that the ordinates Pp, Qq, &c., will be

$$k,\ 2k,\ 3k,\ \&c. \quad . \quad . \quad . \ (1),$$

also from the equation of the curve, $y^2 = 4rx$, the abscissas of the same points will be

$$\frac{k^2}{4r},\ \frac{4k^2}{4r},\ \frac{9k^2}{4r},\ \&c.,$$

and consequently the portions Ap, pq, &c., into which Ax is divided by the ordinates of P, Q, &c., will be

$$\frac{k^2}{4r},\ \frac{3k^2}{4r},\ \frac{5k^2}{4r},\ \&c. \quad . \quad . \quad . \quad . \ (2).$$

Conceive chords AP, PQ, &c. (not in the diagram), to be drawn: then, after the triangle on the base A$p = \frac{k^2}{4r}$, there will be a succession of trapezoids on the several bases (2);

the area of each of which will be found by multiplying half the sum of the two parallel sides by the base of it and by the sine of the angle α at which the axes are inclined, since the product of the base (as pq) by this sine evidently gives the perpendicular distance between the parallel sides. Hence adding the several pairs of parallel sides together—the sides in the series (1), and multiplying each sum by the corresponding base (2) and also by $\sin\alpha$, and dividing by 2, we have (including the triangle A Pp), the following expression for the polygonal space A P Q . . . yx, namely,

$$\text{Area of polygon} = \frac{k^3}{8r}\{1+3^2+5^2 + \ldots\ldots + (2n-1)^2\}\sin\alpha \quad \ldots \quad (3),$$

$$\text{where } k = \frac{y}{n}, \text{ and therefore } k^3 = \frac{y^3}{n^3};$$

$$\text{so that since } y^2 = 4rx,\ \frac{k^3}{8r} = \frac{xy}{2n^3}.$$

Now the next step is to find the sum of the series within the braces in (3): the readiest way of doing this is to first replace the general term $(2n-1)^2$ by its equivalent $4(n-1)n+1$: we shall then have to sum up *two* series of simpler forms, namely, n terms of a series of which the *general* term is $(n-1)n$, and n terms of a series of *units*. By Algebra (p. 174), the first sum is $\frac{(n-1)n(n+1)}{3}$; and the second sum is, of course, n: hence the expression for the series within the braces is $\frac{4n^3-n}{3}$; and consequently (3), substituting $\frac{xy}{2n^3}$ for $\frac{k^3}{8r}$,

$$\text{Area of polygon} = \frac{xy}{2n^3}\left\{\frac{4n^3-n}{3}\right\}\sin\alpha$$

$$= \left\{\frac{2}{3}xy - \frac{xy}{6n^2}\right\}\sin\alpha.$$

Now it is evident that the greater the number n is, the more numerous, and the more minute, do the sides of the polygon become, and the closer does the polygonal line approach to actual coalescence with the curve; which coalescence is attained when the number of sides becomes infinite, but not before; hence, regarding n, in the foregoing expression, to be infinitely great, $\frac{xy}{n}$ must become 0; and the area of the parabolic surface will be

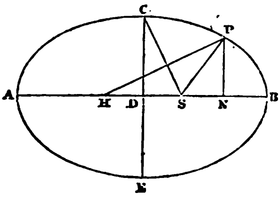
$$\text{Area of parabola } \mathrm{A}yx = \tfrac{2}{3}xy \sin \alpha,$$

that is, the area is equal to two-thirds of the parallelogram Yx; so that the area of any portion of the parabolic surface, cut off by a chord, is always equal to two-thirds of the parallelogram circumscribing that portion.

If a triangle be inscribed in the parabolic segment, the vertex of it being at the extremity of that diameter of the parabola which bisects the base of the triangle, or the chord of the parabolic arc, the area of the segment will be $\frac{4}{3}$ the area of the triangle; hence the segments cut off the parabolic surface, by the other two sides of the triangle, will together be equal to $\frac{1}{3}$ the area of the triangle, or $\frac{1}{4}$ the area of the whole parabolic segment.

Section VIII.—THE ELLIPSE.

An ellipse is a curve in which if any point P be taken, and straight lines be drawn from that point to two certain fixed points S, H, called the *foci* of the ellipse, the *sum* of the two lines PS, PH, shall always be the same; so that if to the two foci the extremities of a cord be fastened, of a length which exceeds the distance SH, and the cord be stretched into a loop by a pencil P, an ellipse will be traced out by moving the pencil round the foci, keeping the cord stretched throughout the motion; for the sum PS+PH of the two *focal distances* of the point P will then remain the same throughout the progress of P.



PROBLEM 1.—*To find the equation of the Ellipse when referred to* AB, CE, *as axes.*

Take D, the middle of the line HS, for the origin of the rectangular axes DB, DC; and let the constant sum of the focal distances of any point P, namely, PH+PS, be represented by $2a$; also let c be put for DS, or DH. Then (x, y) denoting the point P in the curve, we shall have the following conditions, namely,

$$\text{PH}+\text{PS}=2a \quad . \quad . \quad (1), \quad \text{PS}^2=y^2+(x-c)^2 \quad . \quad . \quad (2)$$
$$\text{PH}^2=y^2+(x+c)^2 \quad . \quad . \quad . \quad (3).$$

By addition and subtraction of (2), (3), we have

$$\text{PH}^2+\text{PS}^2=2y^2+2x^2+2c^2, \text{ and } \text{PH}^2-\text{PS}^2=4cx.$$

Dividing this last equation by equation (1), there results

$$\text{PH}-\text{PS}=\frac{2cx}{a},$$

which first added to (1) and then subtracted, gives

$$\text{PH}=a+\frac{cx}{a}, \quad \text{and PS}=a-\frac{cx}{a},$$

$$\therefore \text{PH}^2+\text{PS}^2=2\left(a^2+\frac{c^2}{a^2}x^2\right).$$

But, as just seen,

$$\text{PH}^2+\text{PS}^2=2(y^2+x^2+c^2), \ \therefore y^2+x^2+c^2=a^2+\frac{c^2}{a^2}x^2,$$

$$\therefore a^2y^2+(a^2-c^2)x^2=a^2(a^2-c^2) \quad . \quad . \quad . \quad (4),$$

which is the equation of the curve.

By putting first $y=0$, and then $x=0$, in this equation, we shall ascertain whereabouts the curve crosses the axes; we find that the distances from D are DB $=a$, and DA $=-a$; and DC $=+\sqrt{(a^2-c^2)}$, and DE $=-\sqrt{(a^2-c^2)}$; hence the curve cuts the axis of x at two points B, A, equidistant from D, and the axis of y, at two points C, E, equidistant from D. As a represents DB, let us put b for DC; then $b^2=a^2-c^2$, and therefore, by substitution, the equation of the curve becomes

$$a^2y^2+b^2x^2=a^2b^2, \text{ or } y^2=\frac{b^2}{a^2}(a^2-x^2) \quad . \quad . \quad . \quad (5),$$

or, dividing the first of these forms by $a^2 b^2$,

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}=1 \quad . \quad . \quad . \quad (6)$$

each form being the equation of the ellipse when referred to the diameters (AB, CE) as axes. That each of these lines is entitled to be called a *diameter*, is obvious from the general expressions for y, and x, which expressions are

$$y = \pm\frac{b}{a}\sqrt{(a^2-x^2)}, \text{ and } x = \pm\frac{a}{b}\sqrt{(b^2-y^2)} \quad . \quad . \quad (7)$$

showing that, take whatever value we may for x, from $x = 0$, to $x = \pm a$, there corresponds *two* values of y, equal in length, but opposite in sign; and that, in like manner, take whatever value we may for y, from $y = 0$, to $y = \pm b$, there corresponds *two* values of x, equal in length, but opposite in sign; that is, chords parallel to CE, as well as CE itself, are all bisected by AB; and chords parallel to AB, are all bisected by CE. The curve cannot extend beyond the points A and B, in the direction of the axis of x, nor beyond the points C and E, in the direction of the axis of y; for if in the first of (7) we put $x = \pm a$, the two values of y unite in a single value, namely, $y = 0$; showing that parallels to CE, at the points A and B, never meet the curve again; in other words, that these parallels are *tangents* at the points A and B. In like manner, putting $y = \pm b$, in the second of (7), $x = 0$; showing that parallels to AB are likewise tangents to the curve at the points C and E; thus the four tangents at the points A, B, C, E, entirely enclose the curve, since for a value of x greater than $\pm a$, y is imaginary, and for a value of y greater than $\pm b$, x is imaginary, implying that, beyond the limits mentioned, there does not exist any point belonging to the curve. Not only does the point D bisect the chords AB, CE, but it bisects all the chords drawn through it—the point is therefore the *centre* of the ellipse; for let $y = mx$ be the equation of any straight line through D; substituting this value of y in the equation of the ellipse (5), the corresponding values of x, in the resulting pure quadratic equation, are equal in magnitude, but opposite in sign; and for these, the values of y are also equal and of opposite sign; the two portions of the chord $y = mx$, into which it is divided by D, are therefore equal; that is, all the chords passing through the point D are diameters; and D is the centre of the ellipse.

The greatest of these diameters is AB, and the least is CE, for the length of any semidiameter is

$$\mathrm{D}^2 = x^2+y^2\text{; but } y^2 = b^2-\frac{b^2}{a^2}x^2 \text{ (by equation 5);}$$

$$\therefore\ \mathrm{D}^2 = b^2+\frac{a^2-b^2}{a^2}x^2.$$

Now a^2-b^2 is necessarily positive, for it is equal to c^2 (p. 72), so that the expression for D^2 is greatest when x is greatest; that is, when $x = a$; for which value of x, D is $= \pm a$; hence the points B and A are each at a greater distance from the centre than any other point in the curve.

Again, D is the least possible when x is the least, that is, when $x = 0$; for which value of x, D is $= \pm b$; hence the points C and E are each at a less distance from the centre than any other point in the curve. Hence the greatest diameter of the ellipse is AB, and the least is CE. On this account AB is called the *major diameter* of the ellipse, and CE the *minor diameter*; this pair of diameters is often referred to as the *principal diameters*, or the *principal axes* of the ellipse. The preceding equations, therefore, are so many different forms of the equation of the ellipse when the curve is referred to its principal diameters as axes; the origin being at the centre of the curve.

But there is still another form for the equation, the axes of reference remaining the same; it is immediately deducible from the equation (5); thus, putting a^2-c^2 for its equal b^2, in that equation, it becomes

$$y^2 = \left(1-\frac{c^2}{a^2}\right)(a^2-x^2)\text{, or } y^2 = (1-e^2)(a^2-x^2) \quad . \quad . \quad (8)$$

the symbol e being put for $\frac{c}{a}$; which quantity, namely, the ratio of DS to DB, is called the *excentricity* of the ellipse; that is,

$$e = \sqrt{\frac{a^2-b^2}{a^2}} = \frac{c}{a} = \text{the } \textit{excentricity}.$$

PROBLEM 2.—*To find the equation of the ellipse when the major diameter is taken for the axis of abscissas, and the vertex of that diameter for the origin; as also when one or other of the foci is the origin.*

If the origin of the co-ordinates be removed from the point D to the point A, then to the y of any point in the curve, the new x will be the former x *plus* a; or the former x will be this *minus* a: hence we shall merely have to write $x-a$ for x, in any one of the preceding equations, in order to obtain from it another equation which shall represent the curve in reference to this changed situation of the origin of the axes; thus equation (5) will become changed to

$$a^2y^2+b^2x^2-2b^2ax=0, \text{ or } y^2=\frac{b^2}{a^2}(2ax-x^2) \quad . \quad . \quad . \quad (9),$$

which is the equation of the ellipse when the origin is at the vertex of the major diameter taken for the axis of x. Also in the same circumstances, the equation (8) will become changed to

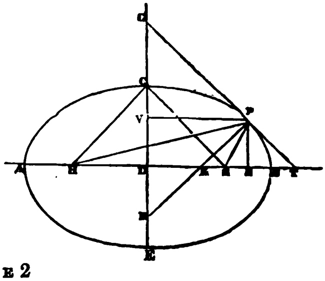
$$y^2=(1-e^2)(2ax-x^2) \quad . \quad . \quad . \quad (10)$$

If the origin were placed at the focus S, then every x in the former equations would have to be increased by ae; that is, by DS or c; and then, instead of (10), the equation (8) would become

$$\begin{aligned} y^2 &= (1-e^2)(a^2-a^2e^2-x^2-2aex) \\ &= (1-e^2)(b^2-x^2-2aex) \quad . \quad . \quad . \quad (11). \end{aligned}$$

If the origin were placed at the other focus H, the equation would be

$$y^2=(1-e^2)(b^2+x^2+2aex) \quad . \quad . \quad . \quad (12).$$



PROBLEM 3.—*To find the polar equation of the ellipse.*

1. Let the centre D be the pole, and the angle PDN $=\theta$; then for any point P in the curve, we have

$x=r\cos\theta$, and
$y=r\sin\theta$,

and therefore from equation (6),

$$r^2\left(\frac{\sin^2\theta}{b^2}+\frac{\cos^2\theta}{a^2}\right)=1,$$

$$\therefore\ r^2=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}=\frac{a^2b^2}{a^2-(a^2-b^2)\cos^2\theta}$$

$$=\frac{b^2}{1-e^2\cos^2\theta}$$

$$\therefore\ r=\frac{b}{\sqrt{(1-e^2\cos^2\theta)}},\ \text{or}\ r^2=\frac{a(1-e^2)}{1-e^2\cos^2\theta}\quad . \quad . \quad . \quad (1),$$

the polar equation of the curve when the centre D is the pole, and DB the fixed axis.

2. Let the focus S be the pole, then SP being r, we have

$$\text{DN}=x=ae+r\cos\theta,\ \text{and}\ y=r\sin\theta\,;$$

$$\text{also (p. 72)},\ r=a-ex,$$

Substituting the preceding expression for x in this expression for r, we have

$$r=a-ae^2-re\cos\theta,\ \therefore\ r(1+e\cos\theta)=a(1-e^2)\,;$$

$$\therefore\ r=\frac{a(1-e^2)}{1+e\cos\theta}\quad . \quad . \quad . \quad (2),$$

which is the polar equation of the curve when the focus S is the pole, and SB the fixed axis.

Equation of the tangent at any point of an ellipse.

Let P be any point in the curve (see preceding diagram), and let x', y', be its co-ordinates. Take a second point in the curve, of which the co-ordinates are x'', y''; then, from the equation of the ellipse (8) p. 74, we have

$$y'^2=(1-e^2)(a^2-x'^2)$$

$$y''^2=(1-e^2)(a^2-x''^2),$$

$$\therefore\ y'^2-y''^2=-(1-e^2)(x'^2-x''^2)\,;$$

$$\text{or}\ (y'+y'')(y'-y'')=-(1-e^2)(x'+x'')(x'-x''),$$

$$\therefore\ \frac{y'-y''}{x'-x''}=(1-e^2)\frac{x'+x''}{y'+y''}.$$

Now the equation of a straight line $Y = mX + h$, through the two points (x', y'), (x'', y''), is

$$Y - y' = \frac{y' - y''}{x' - x''}(X - x'),$$

$$\therefore\ Y - y' = -(1 - e^2)\frac{x' + x''}{y' + y''}(X - x').$$

This line is a *secant*; but if the two points of intersection, by approaching nearer and nearer to each other, unite in a single point (x', y'); that is, if $x' = x''$, and $y' = y''$, the secant will become a tangent, and its equation will be

$$Y - y' = -(1 - e^2)\frac{x'}{y'}(X - x') \quad . \quad . \quad . \ (1).$$

This, therefore, is the equation of the tangent at the point (x', y').

Equation of the Normal.

Since the normal is perpendicular to the tangent, the equation of the normal PR at the point (x', y'), is

$$Y - y' = \frac{y'}{x'(1 - e^2)}(X - x') \quad . \quad . \quad . \ (2).$$

By replacing $(1 - e^2)$ by its value, namely, $\frac{b^2}{a^2}$ (which value might, of course, have been used at the outset), these equations become

Equation of tangent, $Y - y' = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}(X - x') \quad . \quad . \ (3).$

Equation of normal, $Y - y' = \frac{a^2}{b^2} \cdot \frac{y'}{x'}(X - x') \quad . \quad . \ (4).$

From the first of these, we have $a^2Yy' - a^2y'^2 = -b^2Xx' + b^2x'^2$,

$\therefore\ a^2Yy' + a^2Xx' = a^2y'^2 + b^2x'^2 = a^2b^2$ (by the equation of the curve) :

hence another form for the equation of the tangent is

$$a^2y'Y + b^2x'X = a^2b^2, \text{ or } \frac{y'Y}{b^2} + \frac{x'X}{a^2} = 1\,;$$

which differs from the equation of the curve itself only in $y'\mathrm{Y}$ being written for yy, and $x'\mathrm{X}$ for xx.

The equation of the normal, too, may be put in a form which, in certain investigations, is more convenient than that given above: thus, clearing the equation of fractions, and remembering that $a^2-b^2=c^2$, the equation takes the following form, in which X, Y, are replaced by x, y; there being no necessity to retain the former symbols.

$$b^2x'y-a^2y'x+c^2x'y'=0 \quad . \quad . \quad . \quad (5).$$

There is also a peculiar form for the equation of the tangent at a point (x', y') of the curve, which may sometimes be employed to advantage: it is peculiar in this respect—the co-ordinates of the point of contact x', y', do not enter it, but its coefficient of inclination (m), to the axis of x instead. It is obtained thus:

The coefficient of inclination of the line (3) above, is

$$m=-\frac{b^2}{a^2}\cdot\frac{x'}{y'}\text{; so that }\frac{x'}{y'}=-\frac{a^2}{b^2}m\text{;}$$

and we may easily put the equations of both the curve and the tangent, for the point (x', y'), in such a form that x', y', shall enter as the fraction $\frac{x'}{y'}$; and for this fraction we see that $-\frac{a^2}{b^2}m$ may be substituted. These forms are, for the point (x', y') in the *curve*,

$$1+\frac{b^2}{a^2}\cdot\frac{x'^2}{y'^2}=\frac{b^2}{y'^2}\text{; and for the point }(x', y')\text{ in the }\textit{tangent},$$

$$y+\frac{b^2}{a^2}x\cdot\frac{x'}{y'}=\frac{b^2}{y'};$$

or replacing $\frac{x'}{y'}$ by its value given above, these two equations become

$$1+\frac{a^2}{b^2}m^2=\frac{b^2}{y'^2}\text{, and }y-mx=\frac{b^2}{y'}.$$

Hence, multiplying the first of these by b^2, and squaring *the second, to* eliminate y', there results

Equation of the tangent,

$$y-mx=\sqrt{(m^2a^2+b^2)}^* \quad . \quad . \quad . \quad (6).$$

Lengths of the tangent, subtangent, normal, and subnormal.

If in the equation of the tangent (bottom of p. 77) we put $Y=0$, the resulting value of X will be $X=\frac{a^2}{x'}$. This expresses the length DT; and if x' or DN be subtracted from this, we shall have, for the length of the subtangent NT,

$$\text{Subtangent NT}=\frac{a^2-x'^2}{x'}.$$

In like manner, if $Y=0$ be put in the equation of the normal, we shall have for X the length DK; which subtracted from DN or x', will give the length of the subnormal KN. We have therefore to deduce from the equation of the normal the value of $x'-X$ when $Y=0$; and which value is readily found to be

$$\text{Subnormal KN}=\frac{b^2}{a^2}x'=(1-e^2)x',$$

$$\therefore \text{DK}=x'-\text{KN}=e^2x',$$

this latter being the analytical expression for the distance between the centre of the curve and the foot of the normal at the point (x', y'): it is an expression frequently in request.

These values being obtained, those for the lengths of the tangent and normal may be deduced thus. Since $\text{PT}^2=\text{NT}^2+\text{NP}^2$, we have, by substituting the foregoing expression for NT,

$$\text{Tangent PT}=\sqrt{\frac{(a^2-x'^2)^2}{x'^2}+y'^2},$$

$$\text{or, since } y'^2=\frac{b^2}{a^2}(a^2-x'^2),$$

---

* As the coefficient of inclination, m, belongs equally to either of two *parallel* tangents, the sign of the radical here may be either + or −. It is the former when the ordinate of the point of contact is positive, and the latter when the ordinate is negative.

$$\text{Tangent PT} = \sqrt{\left\{\frac{(a^2-x'^2)^2}{x'^2}+\frac{b^2}{a^2}(a^2-x'^2)\right\}},$$

$$= \sqrt{\left\{\left(\frac{a^2-x'^2}{x'^2}+1-e^2\right)(a^2-x'^2)\right\}}.$$

For the length of the normal, we have $\text{PK}^2 = \text{KN}^2 + \text{NP}^2$; which, since $\text{NP}^2 = y'^2 = \frac{b^2}{a^2}(a^2-x'^2)$, by the equation of the curve, is

$$\text{Normal PK} = \sqrt{\left\{\frac{b^4}{a^4}x'^2+\frac{b^2}{a^2}(a^2-x'^2)\right\}},$$

$$= \frac{b}{a}\sqrt{\left\{\frac{b^2-a^2}{a^2}x'^2+a^2\right\}} = \frac{b}{a}\sqrt{(a^2-e^2x'^2)}.$$

At the vertex B of the curve, the normal coincides in direction with BD; but its length is still definite, and is determined by putting $x' = a$ in its equation: we thus get for the length of the normal at the vertex

$$\text{Normal at the vertex B} = \frac{b^2}{a}.$$

And we may here remark that this length is the same as that of the focal ordinate; for the focal ordinate is the radius vector r, when $\theta = 90°$, in which case the expression (2) p. 76, is $\frac{b^2}{a}$, since $1-e^2 = \frac{b^2}{a^2}$. As in the parabola, the double ordinate through the focus is called the *latus rectum*, or *right parameter*: its length is therefore $\frac{2b^2}{a}$; being twice the normal at the vertex B of the curve.

The student will have observed that although *tangent* and *normal* are terms applied to lines of unlimited lengths, so that a point taken on the touching line, however remote from the point of contact this point may be, is still a point on the tangent; and a point anywhere on the perpendicular to this touching line from the point of contact is still a point on the normal, yet the *portion* of the tangent between the point of contact and the prolongation of the major diameter is all that is meant when we speak of the *length* of the tangent; and the portion of the normal between the same point and the major diameter is all that is meant when we speak of the *length* of the normal: in each line the length is limited by the point of intersection of the line with the axis of x. We have just assigned the length of the normal at the vertex of the major diameter, although no one point in it can be regarded as a point of *intersection* with the axis of x, more than any other point in it; in fact, there is no *intersection* at all, but coincidence: how then, it may be asked, can we select a particular point, to the exclusion of all other points common to the axis and the normal, and affirm *that* to be the point which, and which alone, limits the extent of the normal?

The student may satisfy himself that the point thus assigned, is correctly assigned, from the following considerations:

Conceive the point P (diagram, p. 75) to move towards B, and the normal at P, at first PK, to accompany P, still preserving its character as *normal*, till P arrives at B: the intersection K will move along KB, towards B, while the line KP continues closing upon KB; so soon as coincidence takes place, that is, so soon as P arrives at B, the motion of the point K ceases—it will have arrived at its final position, simultaneously with the arrival of PK at its final position: and it is the distance from B of this *last position* of the intersection K, that must correctly be the length of the normal at B. It is plain that that point which is the *last* of the series of points of intersection, must *also* be the *first* of the series of points of coincidence which follows. It is this point of which the distance from B is found above to be $\frac{b^2}{a}$.

These *ultimate* intersections of crossing lines often come under notice in the more advanced departments of analysis, and play an important part in certain enquiries; and it is well that the student should thus early have a clear conception of their existence, and of the strict propriety of the designation; that he should see that if a moveable straight line cross a fixed straight line, and, as it moves along, continuously becomes more and more inclined to the fixed line, though ultimately it entirely coincides with the fixed line, yet that there is, nevertheless, a point in the latter which is necessarily the *last* of the series of intersections, beyond which no point of intersection can exist:—all after *that* is coincidence, no point in which had ever previously been a point of intersection.

We shall now proceed to deduce some properties of the ellipse from the different forms of the equation of the curve here established.

Properties of the Ellipse in relation to its principal Diameters.

THEOREM 1.—The squares of the ordinates are to each other as the products of the segments into which they divide the major diameter.

From equation (5) p. 72, we have

$$\frac{y}{(a+x)(a-x)} = \frac{b^2}{a^2}; \text{ that is, } y^2 : (a+x)\,(a-x) :: b^2 : a^2,$$

which shows that y^2, and the product $(a+x)(a-x)$, of the two parts into which y divides the major diameter $2a$, are always in the same constant ratio, namely, that of b^2 to a^2.

THEOREM 2.—If a circle be described on the major diameter of an ellipse, and ordinates, having a common abscissa, be drawn to both, the ordinate to the ellipse will be to the ordinate to the circle, as b to a.

Let the respective ordinates to the ellipse and circle be y, Y.

The equation of the circle is

$$\text{Y}^2 + x^2 = a^2, \therefore \text{Y}^2 = a^2 - x^2 \quad . \quad . \quad . \quad (1).$$

The equation of the ellipse is

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad . \quad . \quad . \quad (2).$$

Hence, dividing (2) by (1), there results

$$\frac{y^2}{\text{Y}^2} = \frac{b^2}{a^2}, \therefore \frac{y}{\text{Y}} = \frac{b}{a}, \therefore y : \text{Y} :: b : a.$$

THEOREM 3.—In every ellipse described upon the same major diameter the subtangent is always the same for the same abscissa.

This property is deduced at once from the general expression for the subtangent NT, at page 79; for as that *expression* is independent of the value of b, it remains unaltered in value whatever value be given to b, provided that the values of a and x' remain the same.

THEOREM 4.—The normal at any point of an ellipse bisects the angle between the two focal distances of that point.

Let P be any point in the curve (diagram, p. 75), and PR the normal at that point: then S and H being the foci, we have $\text{DS} = \text{DH} = ae$; and putting $\text{Y} = 0$ in the equation (4) of the normal (p. 77), and remembering that $\frac{b^2}{a^2} = 1-e^2$, we have $-1 = \frac{\text{X}-x'}{(1-e^2)x'}$; $\therefore$ $(1-e^2)x' = x'-\text{X}$, $\therefore$ $\text{X} = \text{DK} = e^2x'$ (or see p. 79).

$$\left.\begin{array}{l}\text{Hence } \text{HK} = \text{HD}+\text{DK} = ae+e^2x' = e(a+ex') \\ \quad \text{and } \text{SK} = \text{DS}-\text{DK} = ae-e^2x' = e(a-ex')\end{array}\right\} \quad . \quad . \quad (\text{A}).$$

But $\text{PH}+\text{PS} = 2a$, and $\text{PH}-\text{PS} = 2ex'$ (p. 72, putting e for $\left.\frac{c}{a}\right)$; therefore, by first adding and then subtracting,

$$\text{PH} = a+ex', \text{ and } \text{PS} = a-ex'.$$

Hence, by substitution in (A), we have

$$\text{HK} = e \,.\, \text{PH}, \text{ and } \text{SK} = e \,.\, \text{PS};$$

$$\therefore \frac{\text{HK}}{\text{SK}} = \frac{e \,.\, \text{PH}}{e \,.\, \text{PS}} = \frac{\text{PH}}{\text{PS}},$$

that is, $\text{HK} : \text{SK} :: \text{PH} : \text{PS}$.

Consequently (*Euclid*, 3, vi.) the angle HPS is bisected by the line PK.

Corollary,—It follows from this that the two radii vectores HP, SP, are equally inclined to the tangent at P.

THEOREM 5.—If from the focus a perpendicular to the radius vector of any point in the curve be drawn to meet the tangent at that point in P, the locus of P will be a straight line.

The equation of the line passing through the focus S, or the point $(c, 0)$, and also through the point (x', y') of the curve is

$$y = \frac{-y'}{c-x'}(x-c) \quad . \quad . \quad . \quad (1),$$

and therefore the equation of the perpendicular to it from the point $(c, 0)$ is

$$y = \frac{c-x'}{y'}(x-c) \quad . \quad . \quad . \quad (2),$$

Also the equation of the tangent to the ellipse at the point (x', y') is

$$a^2y'y + b^2x'x = a^2b^2 \quad . \quad . \quad . \quad (3).$$

The equations (2), (3), at the point of intersection, are simultaneous, therefore

$$(c-x')(x-c) = b^2 - \frac{b^2}{a^2}x'x \quad . \quad . \quad . \quad (4),$$

which is an equation of the first degree in x; and therefore represents a straight line: moreover, as y is absent, the locus must be a straight line perpendicular to the major diameter of the ellipse. By solving the equation (4) we find that $x = \frac{a^2}{c} = \frac{a}{e}$; which expresses the distance of the line from the centre of the curve.

By analogy to the parabola, this line is called the *directrix* of the ellipse: there are obviously two directrices; one for each focus. The perpendicular to the radius vector from the focus is called the *polar subtangent* of the curve for the point to which the radius vector is drawn; so that both in the parabola and in the ellipse, the directrix is the locus of the extremities of polar subtangents.

The following are left for the student to investigate for himself:—

6. Prove that the latus rectum is a third proportional to the major and minor diameters.

7. Prove that if a semicircle be described on the major diameter, and a perpendicular to this diameter be drawn, cutting the ellipse in the point P, and the semicircle in P′, the tangents at P and P′, will both meet on the prolongation of the diameter.

8. Prove that in every ellipse the semi-parameter is always greater than once, but less than twice the distance of the focus from the vertex.

9. In the diagram at p. 75, prove that DB . NB = DN . BT, and that DN . NT = AN . NB.

10. In the same diagram, prove that DG . NP = DC^2, and that DV . CG = CV . CD, also that DT . DN = DB^2.

11. Prove that VD is to VR always in the same constant ratio, wherever in the curve the point P may be.

12. Prove also that DR and PN are always in the same constant ratio.

13. The rectangle of perpendiculars from the two foci upon any tangent to the curve is equal to the square of half the minor diameter.

14. The perpendicular from either focus, upon a tangent to the curve, will meet the tangent in a point, the locus of which is the circle described on the major diameter.

15. If pairs of tangents to an ellipse intersect at right angles, the locus of the intersections will be a circle.

Properties of the Ellipse related to its Conjugate Diameters.

The foregoing properties of the ellipse have been deduced from the equation of the curve, expressed in one form or other, and from the equations of the lines drawn in connection with the curve—tangents, normals, &c. All these equations, however, refer the several lines considered to the principal diameters of the curve taken for rectangular axes.

By referring the curve to other systems of co-ordinates, additional properties of interest are likely to unfold themselves. We shall now, therefore, transform the equation $a^2y^2+b^2x^2 = a^2b^2$ into another, equally characterising the ellipse, when the co-ordinate axes, having still the centre for their origin, are *oblique* instead of rectangular. It appears from what is shown at page 56, that in order to transform the equation from rectangular to oblique co-ordinates, without displacing the origin, we must put for x and y, in the primitive equation, the following values, namely,

$$\text{for } x,\ x\cos\alpha+y\cos\alpha';$$
$$\text{and for } y,\ x\sin\alpha+y\sin\alpha'.$$

These substitutions convert the equation

$$a^2y^2+b^2x^2 = a^2b^2$$

into the following, namely :—

$$\left.\begin{array}{r} a^2 \sin^2\alpha' \\ +b^2 \cos^2\alpha' \end{array}\right| y^2 + \left.\begin{array}{r} 2a^2 \sin\alpha \sin\alpha' \\ +2b^2 \cos\alpha \cos\alpha' \end{array}\right| xy$$

$$\left.\begin{array}{r} +a^2 \sin^2\alpha \\ +b^2 \cos^2\alpha \end{array}\right| x^2 = a^2b^2 \quad . \quad . \quad . \quad (\text{A})$$

in which equation α, α' are the angles which the new axes of x and y respectively make with the primitive axis of x; that is, with the major diameter of the ellipse. As these angles are arbitrary, that is, open to any interpretation we may choose to give them, they may be so determined, or the relation between them so fixed, that the following condition may be fulfilled, namely:—

$$a^2 \sin\alpha \sin\alpha' + b^2 \cos\alpha \cos\alpha' = 0 \quad . \quad . \quad . \quad (1),$$

for which condition the middle term will disappear.

Dividing now the left-hand member of this equation by $\cos\alpha \cos\alpha'$, the condition becomes

$$a^2 \tan\alpha \tan\alpha' + b^2 = 0, \text{ or } \tan\alpha' = \frac{-b^2}{a^2 \tan\alpha} \quad . \quad . \quad . \quad (2).$$

This therefore is the relation between α and α' which must subsist in order that the transformed equation (A) may have the same form as the primitive, whatever the inclination α of the new axis of x to the old axis may be; so that, draw this new axis of x, from the origin, in whatever direction we may, the proper direction to be given to the new axis of y, in order that the equation of the curve, when referred to these new axes, may involve only the squares of the variables, will be determined by the condition (2): the transformed equation (A) will then be

$$(a^2 \sin^2\alpha' + b^2 \cos^2\alpha')y^2 + (a^2 \sin^2\alpha + b^2 \cos^2\alpha)x^2$$
$$= a^2b^2 \quad . \quad . \quad . \quad (3).$$

Now this equation of the ellipse is not only of the same form, as respects the *variables*, as the primitive equation, $a^2y^2 + b^2x^2 = a^2b^2$, but it is readily convertible into the same form as respects the *constants* also; for dividing both members of it by a^2b^2, and, for brevity, putting $\frac{1}{b'^2}$ for the coefficient of y^2, and $\frac{1}{a'^2}$ for the coefficient of x^2, the equation takes the form

$$\frac{1}{b'^2}y^2 + \frac{1}{a'^2}x^2 = 1, \therefore a'^2y^2 + b'^2x^2 = a'^2b'^2 \quad . \quad . \quad . \quad (4)$$

which is similar to the primitive equation both as to constants, and as to variables. And consequently, just as it was inferred, from the primitive equation, that a, b, were *rectangular* semi-diameters, so here may it be inferred, in a similar manner, that a', b', are *oblique* semi-diameters of the curve: we have only to put first, $x = 0$, and then $y = 0$, in the equation (4), to assure ourselves that the semi-diameters, which coincide with the axes of y and x respectively, are $y = \pm b'$, and $x = \pm a'$; these being the portions of those axes between the centre and the curve. Hence the diameters, here taken for oblique axes of reference, are $2a'$, $2b'$; and it may be shown (exactly as at p. 78 it was shown from the primitive equation) that each of these diameters bisects all the chords drawn parallel to the other diameter. Pairs of diameters having this property are called CONJUGATE DIAMETERS. Equation (4) is thus the general equation of the ellipse when referred to any pair of conjugate diameters as axes.

Of all pairs of conjugate diameters in an ellipse, the principal diameters are the only pair which are at right angles to one another; for in order that two lines may be perpendicular the one to the other, there must subsist the condition $\tan \alpha \tan \alpha' = -1$; α, α', being the inclinations of the lines to the axis of x: but, for a pair of oblique conjugate diameters, $\tan \alpha \tan \alpha'$ is $= -1$, only when $a = b$, as appears from the condition (2), p. 86; that is, only when the curve is a circle, which is the extreme case of the ellipse, or the case in which the distance between the foci is zero. If, however, in (2) we put $\alpha = 0$, the corresponding value of $\tan \alpha'$ will be infinite, showing that for $\alpha = 0$, α' must be 90°; or that the conjugates must be the principal diameters. From the same condition (2), it further appears that when one of the angles, α, is acute, the other, α', must be obtuse; or that when one of the two tangents is positive, the other must be negative; so that according as one of the conjugate diameters is above, or below, the major diameter, the other conjugate must be to the left, or to the right, of the minor diameter.

Since the equation $a^2y^2+b^2x^2 = a^2b^2$, where a, b, are the principal semi-conjugates, supplied us with a group of properties of the ellipse, it is plain that the equation $a'^2y^2+b'^2x^2 = a'^2b'^2$, having a similar form, will furnish us with a group of like properties extended to conjugate diameters generally. Thus:—

1. Straight lines drawn through the extremities of any diameter, and parallel to its conjugate diameter, are tangents to the curve (p. 73).

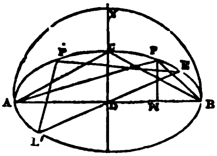
2. The squares of chords drawn parallel to one of two conjugate diameters are as the rectangles of the parts into which they divide the other (p. 82).

3. The rectangle of the ordinate of any point (x', y') in the ellipse, and the ordinate drawn from the centre to the tangent at (x', y'), is always equal to the square of that semi-diameter which is taken for the axis of ordinates (p. 84, Ex. 10).

4. And the rectangle of the abscissa of the point of contact, and the abscissa of the point where the tangent meets the axis of abscissas, is always equal to the square of that semi-diameter which is taken for the axis of abscissas (p. 84, Ex. 10). Not only is the equation of the *curve* when referred to *any* pair of conjugate axes, the same in form as when it is referred to rectangular conjugates, but the equation of the *tangent* is also the same in form (p. 84), the only difference in either equation being that, for the oblique axes, a', b', replaces a, b; that is to say, conjugate diameters in general $(2a', 2b')$, may be employed instead of the particular conjugate diameters $2a$, $2b$, in these equations. Hence, all those properties of the ellipse which are derived exclusively from the equation of the curve, or from the equation of the tangent, or from the combination of these equations, when the axes of reference are the principal conjugate diameters, equally have place whatever conjugate diameters are the axes; the same may be said as to properties in which the sub-tangent is concerned, the sub-tangent being, in general, the intercept of the axis of abscissas between the foot of the ordinate and the point where the tangent meets the axis of abscissas.

We shall now proceed to develope a few additional properties of the ellipse.

Supplemental Chords.—If from the extremities A and B, of the major diameter of an ellipse, two straight lines be drawn to any point P in the curve, these lines are called *supplemental chords*, in reference to that diameter.



The equation of the line BP, passing through the point B, whose co-ordinates, in reference to the principal diameters, are $x = a$, and $y = 0$, is

$$y = m(x-a),$$

and the equation of the line AP, passing through the point A, whose co-ordinates are $x = -a$, and $y = 0$, is

$$y = m'(x+a).$$

Let x', y', be the co-ordinates of the point P; then since this point is in both these lines, and also in the curve, we must have the three following simultaneous equations, namely,

$$y' = m(x'-a) \quad . \quad . \quad . \quad (1)$$

$$y' = m'(x'+a) \quad . \quad . \quad . \quad (2)$$

$$a^2y'^2+b^2x'^2 = a^2b^2 \quad . \quad . \quad . \quad (3)$$

From (1), $m = \dfrac{y'}{x'-a}$; and from (2), $m' = \dfrac{y'}{x'+a}$;

$$\text{therefore } mm' = \frac{y'^2}{x'^2-a^2} \quad . \quad . \quad . \quad (4).$$

And from (3) we have

$$-b^2(x'^2-a^2) = a^2y'^2, \ \therefore \ \frac{y'^2}{x'^2-a^2} = -\frac{b^2}{a^2}.$$

But (4), $\dfrac{y'^2}{x'^2-a^2} = mm'$; therefore $mm' = -\dfrac{b^2}{a^2}$.

If the two meeting chords be drawn from the extremities of any other diameter, they are supplemental chords in reference to *that* diameter; and the relation just established will still have place; for

Let EE' be any diameter whatever, and let x'', y'' be the co-ordinates of the point E in reference to the rectangular conjugates taken as axes; then will $-x''$, $-y''$, be the co-ordinates of the point E'. The equations of the straight lines, EP', E'P', to any point P', or (x', y') will be

$$\text{For EP}' \ldots y-y'' = m(x-x'')$$
$$\text{and for E}'\text{P}' \ldots y+y'' = m'(x+x''),$$

and since the point P′ is in both these lines, we have

$$\left.\begin{aligned} y'-y'' &= m(x'-x'') \\ y'+y'' &= m'(x'+x'') \end{aligned}\right\} \therefore\ mm' = \frac{y'^2-y''^2}{x'^2-x''^2}.$$

But as the same point P′ is also on the curve, therefore

$$\left.\begin{aligned} a^2y'^2+b^2x'^2 &= a^2b^2 \\ \text{and } a^2y''^2+b^2x''^2 &= a^2b^2 \end{aligned}\right\} \therefore\ \frac{y'^2-y''^2}{x'^2-x''^2} = -\frac{b^2}{a^2}.$$

$$\text{Consequently,} \qquad mm' = -\frac{b^2}{a^2} \quad . \ . \ . \quad (5)$$

which is the same condition as before; and implies that whatever diameter of an ellipse be taken, if chords from the extremities of that diameter meet in the curve, the trigonometrical tangents (m, m'), of the inclinations of these chords to the major diameter, always have the constant product (5) above.

PROBLEM.—*To find an expression for the angle contained by supplemental chords drawn to any point* P *in the curve from the extremities of the major diameter.*

From the formula at p. 19 $\tan P = \dfrac{m-m'}{1+mm'}$,

but (p. 89) the expressions for m, m', are

$$m = \frac{y'}{x'-a},\ m' = \frac{y'}{x'+a},$$

$$\therefore\ \tan P = \frac{\dfrac{y'}{x'-a} - \dfrac{y'}{x'+a}}{1+\dfrac{x'^2}{x'^2-a^2}} = \frac{2ay'}{x'^2-a^2+y'^2}.$$

But, since the lines meet in the curve, we must have

$$a^2y'^2+b^2x'^2 = a^2b^2, \text{ or } x'^2-a^2 = -\frac{a^2y'^2}{b^2},$$

hence by substitution,

$$\tan P = \frac{2ay'}{-\dfrac{a^2y'^2}{b^2}+y'^2} = -\frac{2ab^2}{(a^2-b^2)y'}.$$

From this expression it follows that if P be any point of the curve above AB, that is, any point for which y' is positive, the tangent of the angle APB is negative; and therefore the angle itself is *obtuse.* Now the greater any obtuse angle is, the smaller is the tangent of it; the above expression for tan P is the smallest possible when y' is the greatest possible, that is, when $y' = b$. Hence all the angles inscribed in an ellipse, and subtended by the major diameter, are obtuse; and the greatest of these obtuse angles has its vertex at the extremity of the minor diameter; we infer, moreover, that if a semicircle be described on the major diameter, the arc of it will lie wholly *without* the ellipse; for the angles in a semicircle are all right angles; in order, therefore, that an angle subtended by the diameter may be obtuse, the sides of it must meet in a point *within* the semicircle. Further, if an arc of a circle be described passing through the extremities of the major diameter, and also through the vertex of the minor diameter, this arc will lie wholly *within* the ellipse; for the angles inscribed in the circular segment are each of them equal to the *maximum* angle in the semi-ellipse; and consequently are each of them greater than any of the angles in the semi-ellipse, this maximum angle excepted. The values of m and m', for the maximum angle, that is, the angle whose vertex is at the extremity of the minor diameter, are

$$m = -\frac{b}{a},\ m' = \frac{b}{a},\ \text{since when } y' = b,\ x' = 0.$$

It is plain that by changing x into y, and y into x, in the equation of any curve, we merely interchange, or transpose, the axes of co-ordinates. In the ellipse, these changes of the variables give the same equation that we should get by leaving the variables untouched and changing a into b and b into a. Hence making *these* changes, the minor diameter becomes the axis of x, and the major diameter, the axis of y; hence if the angle P be subtended by the minor diameter the expression for its tangent is

$$\tan \text{P} = -\frac{2ba^2}{(b^2-a^2)y'} = \frac{2ba^2}{(a^2-b^2)y'},$$

which is always *positive*; the angle P is therefore always *acute*, and is least when y' is greatest; that is, when it is equal to the semi-major diameter: hence all the angles subtended by the minor diameter are acute; and the least of these acute angles is that whose vertex is at the extremity of the major diameter. The expression for this minimum angle is

$$\tan \text{P} = \frac{2ab}{a^2-b^2};$$

and since this value differs from the former only in its sign, we infer that these two angles are supplements of each other.

It follows, from the fact, that all the angles inscribed in an ellipse, and subtended by the minor diameter are acute—that a semicircle described on the minor diameter will lie wholly within the ellipse; and that if an arc of a circle pass through the extremities of the minor diameter and the vertex of the major diameter, it will lie wholly *without* the ellipse.

THEOREM 1.—The diameters of an ellipse which are parallel to any pair of supplemental chords will always be conjugate diameters.

It appears, by the condition (2) p. 86, that in order that two diameters may be conjugate diameters, their inclinations, α, α' to the major diameter must be such that

$$\tan \alpha \tan \alpha' = -\frac{b^2}{a^2}, \text{ or } m\, m' = -\frac{b^2}{a^2}.$$

putting m, m', for the tangents; and that if this condition have place the diameters cannot be other than conjugate diameters.

Now it has been shown (p. 89), that the inclinations of a pair of supplemental chords, in reference to any diameter, have this relation to each other: hence the diameters parallel to any pair of supplemental chords must be conjugate diameters.

Corollary.—Hence of all pairs of conjugate diameters, those two diameters contain the greatest angle which are respectively parallel to the equal supplemental chords from the extremity of the major diameter. These diameters are also *equal* conjugates; for the acute angles they form with the major diameter are equal.

In relation to these equal conjugate diameters, taken as axes, the equation of the ellipse is simply

$$y^2+x^2=a'^2$$

which, in form, is the same as the equation of the circle; and a circle it would characterise, were it not for this important distinction, namely, that in the equation of the *circle*, the axes of reference are necessarily rectangular; whilst *here*, they are necessarily oblique.

THEOREM 2.—If from the extremities of any diameter $2a'$, supplemental chords be drawn, and they be referred to the oblique diameters $2a'$, $2b'$, as axes, the coefficients of inclination (m, m'), in the equations of these chords, and also, of course, in the equations of the diameters parallel to them, will be such that $m\,m' = -\frac{b'^2}{a'^2}$.

This condition is analogous to that already shown to have place when the principal diameters are taken for axes: it may be proved as follows:—

The equations of the two chords, one passing through the point $(a', 0)$, and the other through the point $(-a', 0)$, are

$$y = m\,(x - a'), \text{ and } y = m'\,(x + a');$$

and by imitating the steps in the investigation of the analogous property, at p. 90, we arrive at the condition announced above.

It is to be observed, however, that the symbols m, m', *here*, do not denote the *tangents* of the angles which the chords make with the axes, since, in the present case, the axes are oblique:—they denote the respective *ratios* of the sines of the two angles which each chord makes with the axes $2a'$, $2b'$, these being oblique; or they each denote the ratio $\frac{x'}{y'}$ of the co-ordinates of any point (x', y') in the chord.

THEOREM 3.—If the tangents be drawn at the extremities of any diameter, and they be limited by two supplemental chords, prolonged beyond their point of intersection, the rectangle of these tangents will always be equal to the square of the diameter parallel to them.

For taking the proposed diameter $2a'$ for the axis of x, and the conjugate to it $2b'$, parallel to the tangents, for the axis of y, and representing the lengths of the two tangents by t, t', it is plain that these lengths will be

$$t = 2ma', \text{ and } t' = 2m'a', \therefore\ t\,t' = 4mm'a'^2.$$

$$\text{But } 4mm' = -\frac{4b'^2}{a'^2} = -\frac{(2b')^2}{a'^2}, \therefore tt' = (2b')^2.$$

Of course, when we have to express the absolute *length* of a line, or the absolute *magnitude* of a surface, any prefixed *sign* is superfluous.

THEOREM 4.—If a point be anywhere taken upon a diameter, and another point be anywhere taken upon the conjugate to that diameter, the rectangle of the two abscissas will be to the rectangle of the two ordinates, as the square of the diameter taken for axis of abscissas, is to the square of the conjugate to it, taken for axis of ordinates.

Let (x', y') be the point taken upon one of the diameters, and (x'', y'') the point taken upon the conjugate diameter; then the coefficients of inclination (m, m') in the equations of these diameters, when referred to any other pair of conjugate diameters, $2a'$, $2b'$, as axes, will be

$$m = \frac{y'}{x'}, \text{ and } m' = \frac{y''}{x''}.$$

But $mm' = -\frac{b'^2}{a'^2}$, $\therefore \frac{y'y''}{x'x''} = -\frac{b'^2}{a'^2}$, which proves the theorem.

THEOREM 5.—The sum of the squares of the reciprocals of any two diameters, at right angles to each other, is a constant quantity, namely, the sum of the squares of the reciprocals of the principal diameters.

Let a_1, b_1, represent any two semidiameters at right angles to one another; it is to be proved that

$$\frac{1}{a_1^2} + \frac{1}{b_1^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

In the general transformed equation at page 86, the axes of reference, originating at the centre of the ellipse, may have any inclinations α, α', whatever to the semidiameter a. If we put, first $y = 0$, and then $x = 0$, in this general equation, there will result, for the squares of the semidiameters coincident with the axes, the values

$$x^2 = \frac{a^2b^2}{a^2 \sin^2\alpha + b^2 \cos^2\alpha}, \quad y^2 = \frac{a^2b^2}{a^2 \sin^2\alpha' + b^2 \cos^2\alpha'}.$$

In these general values for the squares of the two semidiameters, α, α', are arbitrary angles: they are not restricted to any particular values or conditions; we are at liberty to subject them to any condition we please. Let them be chosen so that $\alpha = \alpha' - 90°$, or, which is the same thing, let $\alpha = -(90° - \alpha')$; then by *Trigonometry*,

$$\sin^2\alpha = \cos^2\alpha', \text{ and } \cos^2\alpha = \sin^2\alpha',$$

$$\therefore \frac{1}{x'^2} + \frac{1}{y'^2} = \frac{1}{a_1^2} + \frac{1}{b_1^2}$$

$$= \frac{a^2(\sin^2\alpha + \sin^2\alpha') + b^2(\cos^2\alpha + \cos^2\alpha')}{a^2 b^2}$$

$$= \frac{a^2(\sin^2\alpha + \cos^2\alpha) + b^2(\sin^2\alpha' + \cos^2\alpha')}{a^2 b^2}$$

$$= \frac{1}{b^2} + \frac{1}{a^2}.$$

Let us now return from the oblique conjugates, as axes, to the rectangular. In order to this, we need only find the values of x' and y' in the equations (2) at p. 56. Omitting the constants a and b, if we multiply the first equation by $\sin\alpha'$, the second by $\cos\alpha'$, and subtract the latter result from the former, we shall get x'; and if we multiply the first by $\sin\alpha$, the second by $\cos\alpha$, and subtract in like manner, we shall get y'; and, suppressing the dashes over these letters, shall have

$$\text{For } x,\ \frac{x\sin\alpha' - y\cos\alpha'}{\sin(\alpha' - \alpha)}, \text{ and for } y,\ \frac{y\cos\alpha - x\sin\alpha}{\sin(\alpha' - \alpha)}.$$

These substitutions being made, the equation $a'^2y^2 + b'^2x^2 = a'^2b'^2$ becomes converted into

$$\begin{array}{r|l|l} a'^2\cos^2\alpha & y^2 - 2a'^2\sin\alpha\cos\alpha & \\ b'^2\cos^2\alpha' & \quad -2b'^2\sin\alpha'\cos\alpha' & xy \\ +a'^2\sin^2\alpha & x^2 = a'^2b'^2\sin^2(\alpha' - \alpha) \quad . \quad . & \text{(B)} \\ +b'^2\sin^2\alpha' & & \end{array}$$

This equation, in which $\alpha' - \alpha$ is the angle included between the original oblique axes a', b', applies to *any* system of rectangular diameters taken as axes; but in order that it may be identical to the primitive equation $a^2y^2 + b^2x^2 = a^2b^2$, the coefficients must satisfy the following conditions, namely,

$$a'^2 \cos^2 \alpha + b'^2 \cos^2 \alpha' = a^2 \quad . \quad . \quad . \quad (1)$$
$$a'^2 \sin^2 \alpha + b'^2 \sin^2 \alpha' = b^2 \quad . \quad . \quad . \quad (2)$$
$$a'^2 \sin \alpha \cos \alpha + b'^2 \sin \alpha' \cos \alpha' = 0 \quad . \quad . \quad . \quad (3)$$
$$a'^2 b'^2 \sin^2 [a', b'] = a^2 b^2 \quad . \quad . \quad (4)$$

in which last condition, $[a', b']$ is put to denote the angle between the oblique semidiameters a', b'.

Two interesting properties of the ellipse are implied in these conditions: thus, by adding together (1) and (2), the following Theorem is at once deduced, namely:—

THEOREM 1.—The sum of the squares of the diameters forming any pair of conjugates is always equal to the sum of the squares of the principal diameters.

$$\text{For, } a'^2(\sin {}^2\alpha + \cos {}^2\alpha) + b'^2(\sin {}^2\alpha' + \cos {}^2\alpha') = a'^2 + b'^2$$
$$= a^2 + b^2.$$

THEOREM 2.—Parallelograms circumscribing an ellipse, and having the sides of each parallel to a pair of conjugate diameters, are all equal in surface; the constant area being the rectangle of the principal diameters.

For the area of a parallelogram is equal to the rectangle of two adjacent sides of it multiplied by the sine of the angle between those sides :and from (4) we see that

$$a'b' \sin [a', b'] = ab; \text{ and therefore that}$$
$$4a'b'[\sin a', b'] = 4ab;$$

which proves the theorem.

As already observed, the general equation (B) applies to *any* system of rectangular diameters: whatever rectangular diameters, taken as axes of reference, be chosen, provided only we give the proper special values to α and α', the equation (B) will be that of the ellipse referred to those diameters as axes. If we put first $x = 0$, and then $y = 0$, in the equation (B), we shall get for the resulting values of $\frac{1}{y^2}$, $\frac{1}{x^2}$, the expressions

$$\frac{1}{y'^2} = \frac{a'^2 \cos {}^2\alpha + b'^2 \cos {}^2\alpha'}{a'^2 b'^2 \sin {}^2[a', b']}, \quad \frac{1}{x'^2} = \frac{a'^2 \sin {}^2\alpha + b'^2 \sin {}^2\alpha'}{a'^2 b'^2 \sin^2 [a', b']}.$$

By adding these fractions together, we get $a'^2 + b'^2$ for the sum of the numerators, which sum we have just seen to *be equal to* $a^2 + b^2$; and the common denominator has been shown to be equal to a^2b^2; since a', b', are semiconjugate diameters. Hence, as rather differently proved at page 94,

$$\frac{1}{y'^2}+\frac{1}{x'^2}, \text{ or } \frac{1}{b_1{}^2}+\frac{1}{a_1{}^2}=\frac{a^2+b^2}{a^2b^2}=\frac{1}{b^2}+\frac{1}{a^2}.$$

THEOREM 3.—If through any point within an ellipse chords be drawn; and then at the two extremities of each chord tangents to the ellipse be applied, meeting in a point P, the locus of P will be a straight line. [This Theorem is a generalization of that at page 46; we shall leave it, as also the following Theorems, for the student to prove.]

THEOREM 4.—The rectangle of the radii vectores of any point in an ellipse is equal to the square of the semidiameter, conjugate to that at the point.

THEOREM 5.—Of all pairs of conjugate diameters, the sum of the equal conjugates is the greatest, and the sum of the rectangular conjugates the least.

THEOREM 6.—The radius vector of any point P, is to the perpendicular, SP′, from the focus to the tangent at P, as the diameter ($2b'$), conjugate to that at P, is to the minor diameter ($2b$); that is,

$$\text{SP} : \text{SP}' :: b' : b.$$

THEOREM 7.—The equation of the tangent at a point (x', y') in the curve, and the equation of the chord of contact of two tangents drawn from an external point (x_1, y_1), are of the same forms, respectively, when the axes of reference are oblique conjugate diameters, as when the conjugates are the rectangular diameters; that is,

$$\left.\begin{aligned} a'^2y'y+b'^2x'x=a'^2b'^2, \text{ or } \frac{y'y}{b'^2}+\frac{x'x}{a'^2}=1 \\ \text{or,} \qquad y-mx=\surd(m^2a'^2+b'^2)\,* \end{aligned}\right\}\textit{Equation}$$

of the tangent.

The latter is the equation which a line (whose coefficient of inclination, in reference to conjugate diameters as axes is m) must have, in order that it may touch the curve.

---

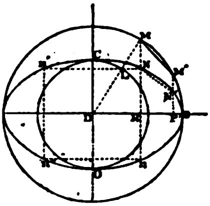
* The coefficient of inclination, m, belongs to each of *two parallel* tangents to the curve; and the radical here is + or −, according as the ordinate of the point of contact is + or −.

$$\left.\begin{aligned} \text{And} \qquad a'^2 y_1 y + b'^2 x_1 x &= a'^2 b'^2 \\ \text{or,} \qquad \frac{y_1 y}{b'^2} + \frac{x_1 x}{a'^2} &= 1 \end{aligned}\right\} \textit{Equation of the}$$

chord of contact of tangents from an external point (x_1, y_1).

Area of the Ellipse.

From the property that if a circle be described upon the major diameter of an ellipse, any ordinate to the ellipse is to the ordinate, for the same abscissa, to the circle, as b to a (p. 82), the area of an ellipse, of which the principal diameters are given, may be readily determined.



In the circle described upon the major diameter of the ellipse, conceive any polygon whatever to be inscribed, one of the sides of which is MM', in the annexed diagram. Draw MP, M'P', perpendiculars to the major diameter of the ellipse, cutting the curve in N and N'; and join N, N'. Let perpendiculars be in like manner drawn from the ends of every side of the polygon inscribed in the circle; and, as here, let the points where they cut the curve be joined; we shall have inscribed in the ellipse a polygon having the same number of sides as that inscribed in the circle.

Let Y and Y' be the ordinates of the points M and M'; and y and y', the ordinates of the points N and N', corresponding respectively to the abscissas x, x'; then the areas of the trapezoids MM'P'P, NN'P'P, are

$$\mathrm{MM'P'P} = \frac{\mathrm{Y}+\mathrm{Y}'}{2}(x-x'), \quad \mathrm{NN'P'P} = \frac{y+y'}{2}(x-x'),$$

$$\therefore \frac{\mathrm{NN'P'P}}{\mathrm{MM'P'P}} = \frac{y+y'}{\mathrm{Y}+\mathrm{Y}'}.$$

Now, page 82, $y = \frac{b}{a}\mathrm{Y}$, and $y' = \frac{b}{a}\mathrm{Y}'$,

$$\therefore \frac{y+y'}{\mathrm{Y}+\mathrm{Y}'} = \frac{\mathrm{NN'P'P}}{\mathrm{MM'P'P}} = \frac{b}{a}.$$

In like manner, each of the trapezoids which compose the polygon inscribed in the ellipse, is to the corresponding trapezoid of the polygon inscribed in the circle as b is to a: hence the sum of all the former trapezoids is to the sum of all the latter, that is, the polygon inscribed in the ellipse, is to the polygon inscribed in the circle, as b is to a.

As this is true however numerous the sides of each polygon may be, or however small each side may be, it must be true in the extreme case when the polygonal lines merge in the curves themselves; consequently if s represents the area of the ellipse, and S the area of the circle, we shall have

$$\frac{s}{S} = \frac{b}{a}; \text{ and therefore } s = \frac{b}{a}S \quad . \quad . \quad . \quad (1).$$

Now the area S, of the circle, is $S = 4a^2 \times {\cdot}7854$; therefore (1) the area s, of the ellipse, is

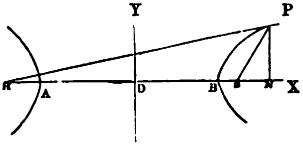
$$\text{Area of the ellipse} = \frac{b}{a}S = 2a \times 2b \times {\cdot}7854 \quad . \quad . \quad . \quad (2).$$

Hence to find the area of an ellipse, we have only to multiply the product of its principal diameters by ·7854.

An ellipse therefore is equal to a circle the square of whose diameter is equal to the rectangle of the principal diameters of the ellipse.

SECTION IX.—THE HYPERBOLA.

The hyperbola is a curve such that if from any point P in it, two lines be drawn to two fixed points S, H, called the *foci* of the curve, their difference PH − PS, will be always the same. In this curve, as in the ellipse, the focal distances PS, PH, of any point P, are called the *radii vectores* of the point.



PROBLEM 1.—*To find the equation of the hyperbola, when referred to* DX, DY, *as rectangular axes, the origin being at* D, *the middle point of the line* HS.

Let any point P in the curve be denoted by (x, y); and let $2a$ be put for the constant difference PH−PS; put also

c for DS, or DH: then the equation of the locus of P is to be deduced from the following conditions, namely,

$$\text{PH}-\text{PS}=2a\ .\ .\ .\ (1);\qquad \text{PH}^2=y^2+(x+c)^2\ .\ .\ .(2);$$
$$\text{PS}^2=y^2+(x-c)^2\ .\ .\ .\ (3).$$

Now in order to eliminate PH, PS, from these equations, take the sum and difference of (2) and (3): we thus have

$$\text{PH}^2+\text{PS}^2=2(y^2+x^2+c^2)\ .\ .\ .\ (4);$$
$$\text{PH}^2-\text{PS}^2=4cx\ .\ .\ .\ (5)$$

and dividing (5) by (1), $\text{PH}+\text{PS}=\frac{2cx}{a}$: hence first adding (1) to this, and then subtracting, $\text{PH}=\frac{c}{a}x+a$, and $\text{PS}=\frac{c}{a}x-a$; which values substituted in (4), convert that equation into

$$\frac{c^2}{a^2}x^2+a^2=y^2+x^2+c^2,\text{ or }a^2y^2+(a^2-c^2)x^2$$
$$=a^2(a^2-c^2)\ .\ .\ .\ (6),$$

which being the general relation between the x and y, of any point in the curve, is one form of the equation of the curve.

If in this equation we put $y=0$, we shall ascertain the points where the curve crosses the axis of x: the abscissas of these points are therefore $x=+a$, and $x=-a$; so that the curve crosses HS in two points B, A, equidistant from D; and since the points S, H, are also equidistant from D, it follows that BS=AH. If we put $x=0$ in the equation (6), we find that, for the corresponding values of y, we have the expression

$$y=\sqrt{(a^2-c^2)}=\sqrt{(c^2-a^2)}\sqrt{-1}.$$

showing that there is no ordinate to the curve at the origin; in other words, that the curve has no intersection, or contact, with the axis of y: that a^2-c^2 is *negative*, or that a is less than c, is plain; for a is DB, as just shown, and c is DS. Although the ordinate at the origin is thus seen to be *imaginary*, that is, having no real existence as an ordinate *to the curve*, yet if we mark off from the axis of y a length, *measured from* the origin D, equal to the real quantity

$\surd(c^2-a^2)$, and put b for this length, the equation (6) will assume a form analogous to the equation of the ellipse; for it will then be

$$a^2y^2-b^2x^2=-a^2b^2, \text{ or } \frac{y^2}{b^2}-\frac{x^2}{a^2}=-1, \text{ or}$$

$$y^2=\frac{b^2}{a^2}(x^2-a^2) \quad . \quad . \quad . \quad (7)$$

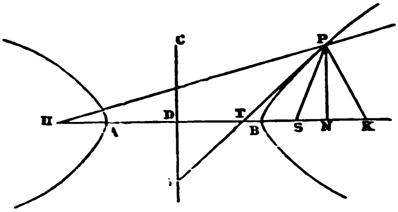
and therefore,

$$\left.\begin{aligned} y&=\pm\ \frac{b}{a}\ \surd(x^2-a^2)\,; \\ x&=\pm\ \frac{a}{b}\,\surd(y^2+b^2) \end{aligned}\right\} \quad . \quad . \quad (8).$$

From the first of the equations (8) we see that for all values of x between the limits $x=+a$, and $x=-a$, the values of y are imaginary; in other words, that between these limits, which the points B and A terminate, perpendiculars to the line AB never meet the curve; but that beyond these limits, in both directions, however great the value of x may be, there are always *two* real and equal values for the corresponding y; one indicating a point $(\pm x, y)$, on the curve, and the other, the opposite point $(\pm x, -y)$; the former above, and the latter below the axis of x: the double sign $\pm x$, implying that the inference is the same whether the abscissa be measured in the positive, or in the negative, direction. We conclude, therefore, that the curve consists of two distinct branches; one branch spreading out towards the right, and the other, equal to it, spreading out towards the left; and that these branches are separated from one another by the interval which lies between the two parallels whose equations are $x=a, x=-a$; that is, between the two perpendiculars to AB, drawn from the extremities A, B, of this line; and which are tangents to the curve at the points A, B.

It thus appears that no point in the hyperbola can be situated anywhere between these parallel tangents, but that on each side of them the branches of the curve extend onwards without termination, and that to each one of equal and opposite abscissas, there belongs a pair of equal and opposite ordinates; and also that all four of the ordinates, belonging to equal and opposite abscissas, are equal in length. On this account we may conclude that chords drawn parallel to AB are all bisected by YD and its prolongation; and that chords parallel to Y D are all bisected by the prolongation of A B.

Let any straight line $y=mx$, drawn through the origin D, meet the curve in a point (x', y'): it will also meet the curve in the point $(-x', -y')$, situated in the opposite branch, since for $y=mx$, in (8), there are two values of x, equal in magnitude, but opposite in sign, and therefore two values of y, namely $y=mx$, equal and opposite in sign: hence the straight line through D meets the curve in two points, on opposite branches, and is *bisected* in the point D, because $x^2+y^2=(-x)^2+(-y^2)$. The point D, thus bisecting all chords passing through it, is the *centre* of the hyperbola, and the chords themselves *diameters*; the chord A B, or $2a$, joining the vertices of the two branches of the curve, is called the *transverse*, or *principal* diameter; and that portion D C of the axis of y, of which the length is $DC=\sqrt{(c^2-a^2)}$, though the line is unconnected with the curve, is called the principal *second* semidiameter of the hyperbola; or taking $DE=DC$, $CE=2\sqrt{(c^2-a^2)}=2b=$principal *second* diameter.



These could not be called major and minor diameters, as in the ellipse, because b may *exceed* a, since c may be any length greater than a in $\sqrt{c^2-a^2}$).

The equation (7) is thus the equation of the hyperbola in reference to its principal diameters $2a$, $2b$. If the particular curve be such that $a=b$, the hyperbola would be what is called an *equilateral* hyperbola, because of this equality of its principal diameters; so that while equality of the principal diameters converts the ellipse into a circle, the same equality converts the hyperbola into an equilateral hyperbola.

As in the ellipse, the ratio $\frac{c}{a}=e$, is called the *excentricity*;

the expression for c *here*, is $c^2 = a^2 + b^2$; whilst in the ellipse, it is $c^2 = a^2 - b^2$: and it is obvious that, in general, since the equation of the hyperbola differs from that of the ellipse only in the sign of b^2, the various analytical results arrived at in the discussion of the ellipse, must become applicable to the hyperbola when $+b^2$, in those results, is changed into $-b^2$.

If the origin of the rectangular axes be removed from D to B, as it will be if $x+a$ be substituted for x in (7), the equation of the curve will be

$$y^2 = \frac{b^2}{a^2}(x^2 + 2ax) \quad . \quad . \quad . \quad (9).$$

In the case of the ellipse (page 75), the origin was removed the distance a to the *left*, and the substitution was therefore $x-a$ for x; here, the removal is to the *right*; so that to assimilate the two cases, as to transference of origin, the equation (9), at page 75, should be $y^2 = -\frac{b^2}{a^2}$ (x^2+2ax); which is converted into (9) above, by changing the sign of b^2.

In terms of the excentricity e, the equation (7) of the hyperbola is

$$y^2 = (c^2-1)(x^2-a^2), \textit{ equa. in terms of the excentricity } (10),$$

because $b^2 = c^2 - a^2$, and $c^2 = a^2e^2$; giving e^2-1 for $\frac{b^2}{a^2}$. In the ellipse, the corresponding equation is $y^2 = (1-e^2)$ (a^2-x^2); so that the two equations are apparently identical without any change of sign; but the identity is in appearance only, since in the ellipse $1-e^2$ stands for $\frac{b^2}{a^2}$, whilst in the hyperbola it stands for $-\frac{b^2}{a^2}$: in the ellipse therefore e is always *less* than 1: in the hyperbola e is always *greater* than 1. In each curve, the line $x = \frac{a}{e}$ is the *directrix*.

For the equation in terms of the excentricity, when the origin is at B, we have only to substitute e^2-1 for

$\frac{b^2}{a^2}$ in (9), and the form becomes

$$y^2 = (e^2-1)(x^2+2ax) \quad . \quad . \quad . \ (11).$$

And if the origin be at the focus S, since DN (the primitive x) $=$ SN$+ae = x+ae$, this is the value which must replace x in (9) instead of $x+a$: making therefore this substitution, the equation (9) becomes

$$y^2 = (e^2-1)(x^2+2aex+b^2) \quad . \quad . \quad . \ (12),$$

the equation of the curve when the origin is at the focus S.

Put $x = 0$ in this last equation, and there results for y, the ordinate at the focus,

$$y = \frac{b}{a}.b; \text{ that is, } y = \frac{b^2}{a}, \therefore 2y = \frac{2b^2}{a} = \frac{(2b)^2}{2a},$$

so that (like as in the ellipse) the double ordinate at the focus is a third proportional to the transverse and second (or conjugate) diameters.

This double ordinate, as in the ellipse, is called the *principal parameter* (p), or the parameter of the transverse diameter; and hence (equation 7), when the origin is at the centre,

$$y^2 = \frac{p}{2a}(x^2-a^2) \text{ is the equation in terms of the principal parameter} \quad . \quad . \quad . \ (13).$$

In the equilateral hyperbola, $b = a$; so that then $p = 2a$; that is, the principal parameter is equal to the transverse diameter.

If the origin of the axes be at B, then putting $\frac{p}{2a}$ for $\frac{b^2}{a}$ in (9), the equation is

$$y^2 = \frac{p}{2a}+px \quad . \quad . \quad . \ (14).$$

PROBLEM 2.—*To find the Polar Equation of the Hyperbola.*

1st. Let the centre be the pole: then substituting

$r \cos \theta$ for x, and $r \sin \theta$ for y, in the equation (10) of the curve, the expression for the radius vector r, will be

$$r = a\sqrt{\frac{e^2-1}{e^2\cos^2\theta-1}} \text{ (\textit{centre} for pole)} \quad . \quad . \quad . \quad (1).$$

2nd. Let the focus H be the pole; then, HP or $r = ex+a$ (page 100); but x, measured from the centre, $= r\cos\theta - ae$: hence by substitution,

$$r = er\cos\theta - ae^2 + a, \ \therefore \ (1-e\cos\theta)r = a(1-e^2)$$

$$\therefore \ r = \frac{a(1-e^2)}{1-e\cos\theta} \text{ (\textit{focus} H for pole)} \quad . \quad . \quad . \quad (2).$$

3rd. Let the focus S be the pole: then SP or $r = ex - a$, and $x = r\cos\theta + ae$; therefore by substitution,

$$r = er\cos\theta + ae^2 - a, \ \therefore \ (1-e\cos\theta)\, r = a(e^2-1)$$

$$\therefore \ r = \frac{a(e^2-1)}{1-e\cos\theta} \text{ (\textit{focus} S for pole)} \quad . \quad . \quad . \quad . \quad (3).$$

In each of these three polar equations the angle θ has its opening to the *right*, that is, the initial direction of the radius vector coincides with the positive direction of the axis of x: but if in the last case the supplement θ' of θ be employed instead of θ, then since $\cos\theta = -\cos\theta'$, the equation will be

$$r = \frac{a(e^2-1}{1+e\cos\theta'} \quad . \quad . \quad . \quad (4).$$

It may be noticed here, as a property common to all the *conic sections*, as the three curves—the ellipse, the hyperbola, and the parabola, are called, that a straight line, in whatever direction it be drawn, cannot cut the curve in more points than two: this arises from the circumstance that the equations of these curves are each of only the second degree, and the equation of a straight line of only the first degree. If the expression for y in this latter equation be substituted for that variable in the equation of any one of the curves, the result will be an equation of the second degree in x; which variable therefore can have but *two* values, so that the points of intersection of the straight line and curve—of all of which points the values of x in the equation of the second degree are the abscissas, cannot exceed two in number.

We learn from this that a chord in one branch of the hyperbola can never, by being prolonged, meet the other branch; and that a chord uniting the two branches can never, by being prolonged, meet either branch again.

In the equation (2) above, the initial direction of the radius vector is the direction HB, and B is the point in the curve to which the r in that equation applies when $\theta = 0$, that is, when $\cos\theta = 1$; for this value of θ, the equation gives $r = a(1+e) = a+ae =$ HB. As θ increases, by the revolution of r about the focus H, r continues to denote the radius vector of a point in this further branch of the curve, till θ arrives at a value such that the prolongation of r meets the nearer branch, or the branch within which the focus H lies: as shown above, a line from H, for *this* value of θ, cannot meet the opposite branch of the curve; the r, therefore, then, and henceforward so long as r is part of a chord of the near branch, belongs to a point in that branch only: thus, put $\theta = 90°$; then (2), $r = a(1-e^2) = -\frac{b^2}{a}$, which we know to be the negative ordinate at the focus H (p. 104).

We shall again advert to this subject when we come to speak of *asymptotes.*

Equation of the Tangent at any Point of the Hyperbola.

Let (x', y') and (x'', y'') be any two points upon one of the branches of an hyperbola: the equation of a straight line passing through them is

$$y-y' = \frac{y'-y''}{x'-x''}(x-x') \quad . \quad . \quad . \quad (1).$$

And since the points belong to an hyperbola, their coordinates must fulfil the conditions

$$a^2y'^2-b^2x'^2 = -a^2b^2 \quad . \quad . \quad . \quad (2),$$
$$a^2y''^2-b^2x''^2 = -a^2b^2 \quad . \quad . \quad . \quad (3).$$

Subtracting the third of these simultaneous equations from the second, we have

$$a^2(y'+y'')(y'-y'')=b^2(x'+x'')x'-x''),$$

$$\therefore \frac{y'-y''}{x'-x''}=\frac{b^2}{a^2}\cdot\frac{x'+x''}{y'+y''}.$$

Hence by substitution in (1) the equation of the secant is

$$y-y'=\frac{b^2}{a^2}\cdot\frac{x'+x''}{y'+y''}(x-x').$$

Let now the two points (x', y'), (x'', y''), by approaching closer and closer, unite in the single point (x', y'); the secant will then become a tangent at (x', y'); its equation will therefore be

$$y-y'=\frac{b^2}{a^2}\cdot\frac{x'}{y'}(x-x') \quad . \quad . \quad . \quad (4),$$

$$\text{or } a^2y'y+b^2x'x=-a^2b^2,$$

so that, as in the ellipse, the equation of the tangent at a point (x', y') may be derived from the equation of the curve, by substituting $y'y$ for y^2, and $x'x$ for x^2.

The above equation, as might have been anticipated, differs from that for the tangent to an ellipse only in the sign of b^2; and without any independent investigation, such as that here given, we might have safely deduced the equation of the tangent to the hyperbola from that to the ellipse by simply writing $-b^2$ for $+b^2$. And in this way, without repeating the steps at page 78, we may at once write the equation of the tangent in the following different form, namely,

$$y-mx=\surd(m^2a^2-b^2).$$

From (4), the equation of the normal at the point (x', y') of the curve is

$$y-y'=-\frac{b^2}{a^2}\cdot\frac{x'}{y'}(x-x'),$$

and from these equations the expressions for the lengths of the tangent, subtangent, normal, and subnormal, may be deduced exactly as in the case of the ellipse at page 79. These lengths are as follow:

$$\text{Tangent PT} = \sqrt{\left\{\frac{(x'^2-a^2)^2}{x'^2} + \frac{b^2}{a^2}(x'^2-a^2)\right\}}$$

$$= \sqrt{\left\{\left(\frac{x'^2-a^2}{x'^2}+e^2-1\right)(x'^2-a^2)\right\}}.$$

$$\text{Normal PK} = \frac{b}{a}\sqrt{(e^2x'^2-a^2)}.$$

$$\text{Subtangent NT} = \frac{x'^2-a^2}{x'}.$$

$$\text{Subnormal KN} = \frac{b^2}{a^2}x' = (e^2-1)x',\ \therefore\ \text{DK} = x'+\text{NK} = e^2x'.$$

This expression for DK, the distance between the centre and the foot of the normal at the point (x', y'), should be borne in remembrance: it is the same expression whether the curve be the hyperbola or the ellipse (see page 79).

Properties of the Hyperbola in reference to its Principal Diameters.

Theorem 1.—The squares of the ordinates are to each other as the products of the parts of the prolonged principal diameter, included between those ordinates and the two vertices of the curve.

The proportion at page 82 (Theorem 1), applies to the hyperbola, when b^2 is changed into $-b^2$; or leaving b^2 unchanged, when $x-a$ is substituted for $a-x$: therefore, in the hyperbola,

$$y^2 : (a+x)(x-a) :: b^2 : a^2;$$

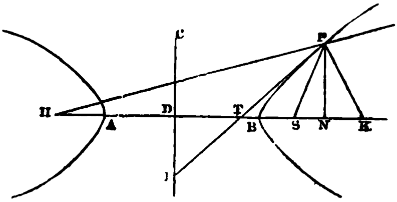
that is, y^2, and the products of the distances $(a+x)$, and $(x-a)$, of y from the two vertices, are always in the constant ratio of b^2 to a^2.

When $b=a$, that is, in the equilateral hyperbola, $y^2 = (a+x)(x-a)$; so that then the property is analogous to that of the circle.

Theorem 2.—In different hyperbolas having a common transverse diameter, the subtangent for the same abscissa is always the same. For the expression for the subtangent, being independent of b, continues to be the same whatever be the length of b, provided only that a and x' remain the same.

THEOREM 3.—The normal at any point of an hyperbola bisects the exterior angle formed by one radius vector of the point and the prolongation of the other: and the two radii are equally inclined to the tangent at the point.

By the fundamental property of the curve, and the deductions from it (p. 100), we have



$$\text{PH} - \text{PS} = 2a,$$
$$\text{and PH} + \text{PS} = 2ex';$$

e being put for $\frac{c}{a}$ in the expressions referred to. By first adding, and then subtracting, we have

$$\text{PH} = ex' + a, \text{ and PS} = ex' - a.$$

Also, $\text{SK} = \text{DK} - \text{DS} = e^2x'$[*] $- ae = e(ex' - a) = e.\text{PS}$

$$\text{HK} = \text{DK} + \text{HD} = e^2x' + ae = e(ex' + a) = e.\text{PH}$$

$$\therefore \frac{\text{HK}}{\text{SK}} = \frac{\text{PH}}{\text{PS}} \therefore \text{HK} : \text{SK} :: \text{PH} : \text{PS}.$$

Therefore PK bisects the exterior angle formed by PS and the prolongation of HP. (Euclid 4, VI.).

The focal distances KP, HP, are equally inclined to the tangent; that is, the angles SPT, HPT, are equal; for KP is at right angles to the tangent; and the former equal angles being taken from the equal right angles, leaves the equal angles HPT, SPT; since HPT is equal to the opposite vertical angle.

PROBLEM.—*To find an expression for the tangent of the angle between two lines drawn from any point in the curve of the hyperbola to the extremities of the transverse diameter.*

It was shown at p. 90, that an angle (P) at any point of an ellipse, and subtended by the diameter $2a$, had for its tangent the value—

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[*] Page 108.

$$\tan P = \frac{-2ab^2}{(a^2-b^2)y'}, \text{ in the ellipse.}$$

$$\therefore \tan P = \frac{2ab^2}{(a^2+b^2)y'}, \text{ in the hyperbola.}$$

As this expression is independent of x', it must be the same for the point $(-x', y')$, as for the point $(+x', y')$. Moreover, when y' is positive, that is, for every point P above the axis of x, the expression for tan P is positive: hence all the angles at the curve, which are subtended by the transverse diameter, are acute: they diminish as y' increases, and become 0 when y' is infinite; that is, when the point is infinitely remote. But as y' diminishes, that is, as the point P approaches nearer and nearer to the extremity of the transverse diameter, the tangent of the angle at P approaches nearer and nearer, in magnitude, to infinity; and therefore the angle itself, to a right angle: hence the acute angle may be of any magnitude whatever within the entire rangle of acute angles, that is, within the extreme limits of 90° and 0°. As in the ellipse, the chords from the extremities of the transverse diameter, meeting in a point of the curve, are called *supplemental chords.*

The tangents of the angles which these supplemental chords make with the axis of x, being m, and m', we find, as in the ellipse (p. 89), that

$$m = \frac{y}{x+a}, \text{ and } m' = \frac{y}{x-a}, \therefore mm' = \frac{y^2}{x^2-a^2}.$$

At the point (x', y'), common to these meeting lines and the curve, we must have, from the equation of the hyperbola,

$$\frac{y'^2}{x'^2-a^2} = \frac{b^2}{a^2}; \therefore mm' = \frac{b^2}{a^2} \quad . \quad . \quad . \quad (1),$$

$$\text{or } m' = \frac{b^2}{a^2m} \quad . \quad . \quad . \quad (2),$$

that is, the product of the trigonometrical tangents of the two angles formed by the transverse diameter prolonged to the right, and any pair of supplemental chords, is constant, being always equal to $\frac{b^2}{a^2}$. As this value is positive, we infer that the two angles are always either both acute, or both obtuse; *acute* when the point in which the supplemental chords meet is on the right-hand branch of the curve, and *obtuse* when it is on the left-hand branch. If the hyperbola be equilateral, that is, if $b = a$, then $mm' = 1$, or $m' = \frac{1}{m}$; so that the tangent of one of the angles is the co-tangent of the other: the two angles together must therefore make a right angle; when they are both acute, or when the point is on the right-hand branch; and they must together make three right angles when they are both obtuse, or when the point is on the left-hand branch.

Properties of the hyperbola in reference to its conjugate diameters.

By transforming the equation of the hyperbola from rectangular axes to oblique, the origin in each system being at the centre of the curve, we shall find, exactly as at p. 85, for the ellipse, that the general transformed equation will be,

$$\left.\begin{array}{r} a^2\sin^2\alpha' \\ -b^2\cos^2\alpha' \end{array}\right| y^2 + \left.\begin{array}{r} 2a^2\sin\alpha\cos\alpha' \\ -2b^2\cos\alpha\cos\alpha' \end{array}\right| xy$$

$$\left.\begin{array}{r} +a^2\sin^2\alpha \\ -b^2\cos^2\alpha \end{array}\right| x^2 = -a^2b^2$$

in which equation α, α', denote the angles which the oblique co-ordinates x, y, make with the primitive axis of x. In order that the term containing xy may disappear from this equation, the angles α, α', must be so related that

$$a^2\sin\alpha\cos\alpha' - b^2\cos\alpha\cos\alpha' = 0,$$

or, dividing by $\cos\alpha\cos\alpha'$,

$$a^2\tan\alpha\tan\alpha' - b^2 = 0, \text{ or, } \tan\alpha' = \frac{b^2}{a^2\tan\alpha} \quad . \quad . \quad . \quad (1).$$

The relation therefore between the two angles must be such that, one of them being chosen at pleasure, the other must satisfy this condition; and this is the only restriction necessary: hence an infinite number of pairs of oblique axes exist, each of which will render the equation of the curve, when referred to that pair, of the form

$$(a^2 \sin^2 \alpha' - b^2 \cos^2 \alpha')y^2 + (a^2 \sin^2 \alpha - b^2 \cos^2 \alpha)x^2$$
$$= -a^2b^2 \quad . \quad . \quad . \quad (2),$$

or which is the same thing,

$$(a^2 \sin^2 \alpha' - b^2 \cos \alpha')y^2 - (b^2 \cos^2 \alpha - a^2 \sin^2 \alpha)x^2$$
$$= -a^2b^2.$$

Now dividing this equation by a^2b^2, and then putting

$$a'^2 \text{ for } \frac{a^2b^2}{b^2 \cos^2 \alpha - a^2 \sin^2 \alpha}$$

$$\text{and } b'^2 \text{ for } \frac{a^2b^2}{a^2 \sin^2 \alpha' - b^2 \cos^2 \alpha'},$$

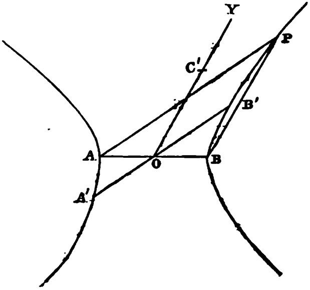
it will take the form

$$\frac{1}{b'^2}y^2 - \frac{1}{a'^2}x^2 = -1,$$

$$\text{or } a'^2y^2 - b'^2x^2 = -a'^2b'^2 \quad . \quad . \quad . \quad (3),$$

which is the same as that of the primitive equation of the curve, where the axes are rectangular instead of oblique (as here) and coincide with the principal diameters—a conclusion we might have arrived at by simply changing b'^2 for $-b'^2$, in the corresponding transformed equation of the ellipse (p. 87).

Since for $x = 0$ in (3), we have $y = \pm\sqrt{-b'^2}$
$= \pm b'\sqrt{-1}$,
and for $y = 0$ in (3), we have $x = \pm a'$,



it follows 1st that it is impossible for the curve to meet the new axis of y; and 2nd, that it crosses the axis of x at two points $x = a'$, $x = -a'$, equidistant from the centre O. As y is 0 at each of these points (B' and A'), we infer that parallels to the axis of y, through B' and A', must be *tangents* to the curve at those points.

It also follows from (3), that for every x, whether positive or negative, which, in absolute magnitude, *exceeds* a', there are *two* values of y, equal but of opposite sign; and that for every x, which, in absolute magnitude, is *less* than a', the corresponding y is imaginary; that is a parallel to OY, drawn through any point between A′ and B′, never meets the curve, the two branches of which therefore wholly lie, one to the right, and the other to the left, of the interval between the tangents to the curve at A′ and B′.

Since there are two equal and opposite ordinates for every abscissa to the right which exceeds OB′, and for every abscissa to the left which exceeds OA′, it follows that the axis of x bisects all the chords drawn parallel to the axis of y. And, moreover, since from the same equation (3) there are two equal and opposite values of x for every value of y, the axis of y must bisect all the chords drawn parallel to the axis of x:—these latter chords unite the two distinct branches of the curve; they all traverse the separating interval between them. A length equal to b', measured from O on the axis of y, is called the *second* diameter to the diameter A′B′; the two together are *oblique conjugate diameters*; and the equation (3), without any restriction as to whether the conjugates are rectangular or oblique, is the *general* equation of the hyperbola in reference to a pair of conjugate diameters ($2a'$, $2b'$), taken as axes, the origin being at the middle of $2a'$. We see from the conditions (2) and (1) pp. 110, 111, that a pair of diameters drawn parallel to a pair of supplemental chords are always *conjugate* diameters; and of every pair of these, as we have just seen, one only is a *transverse* diameter, that is, one only has its extremities on the curve: the other is a *second diameter*: it never meets the curve however far it be prolonged. And as we have seen that the angle P between supplemental chords diminishes more and more, as P becomes more and more remote, approaching continuously to zero, it follows that the angle between a pair of conjugate diameters, in like manner, approaches nearer and nearer to evanescence, or the diameters themselves nearer and nearer to coalescence, as B′, the extremity of the transverse, becomes more and more remote; and that there is no limit to this approximation to actual coincidence.

Just as at p. 110, it was shown that $mm' = \frac{b^2}{a^2}$, so may it with the transverse, must ultimately coalesce with it, the angle between them having gradually diminished down to zero.

This is a peculiarity that distinguishes the hyperbola from the ellipse in a very marked manner, and thus implies certain properties of the former curve to which those of the ellipse bear no analogy, although most of the other properties are common to both curves, or at least are strictly analogous. We shall now proceed to investigate these peculiar properties of the hyperbola.

The equation of the hyperbola, referred to its principal diameters as axes, being

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \text{ we have } y = \pm\frac{b}{a}\sqrt{(x^2 - a^2)}, \text{ or}$$

$$y = \pm\frac{b}{a}x\sqrt{(1 - \frac{a^2}{x^2})}.$$

Now the more x increases, the more the fraction $\frac{a^2}{x^2}$ diminishes; so that the corresponding values of y approach nearer and nearer to the extreme values $y = \pm\frac{b}{a}x$, which however cannot be actually attained, since they imply a value infinitely great for x. Of the equations

$$y = \pm\frac{b}{a}x \quad . \quad . \quad . \quad (1), \quad y = \pm\frac{b}{a}x\sqrt{(1 - \frac{a^2}{x^2})} \quad . \quad . \quad . \quad (2)$$

the first is that of a straight line drawn from the centre of the curve represented by the second equation; and since, for the same abscissa, the difference between the ordinate of the straight line and the ordinate of the curve becomes less and less, as the abscissa increases in magnitude, and vanishes altogether when the abscissa becomes infinite in length, but not till then, we conclude that the straight line (1) approaches nearer and nearer to the curve (2) the further it is prolonged from the origin, or centre of the curve; and that although the interval between the straight line and the curve (in the direction of y) becomes eventually less than any finite interval that can be assigned—however small this may be, yet it can never wholly disappear; that is, the straight line can never actually meet *the curve.*

Hence, if through the origin O two straight lines KL, MN, be drawn, of which the equations are

$$y = \frac{b}{a}x, \text{ and } y = -\frac{b}{a}x,$$

that is, such that

$$\tan \text{KOB} = \frac{b}{a}, \text{ and } \tan \text{MOB} = -\frac{b}{a} \quad . \quad . \quad . \quad (3),$$

the latter angle being the supplement of the former, these two lines will continually approach nearer and nearer to the curve the further they are prolonged, and yet can never meet the curve within any finite distance from O, in whichever direction from O that distance be measured; that is, whether the supposed point of meeting be (x, y), or $(-x, y)$, or $(x, -y)$, or $(-x, -y)$. These two lines are called the *asymptotes of the hyperbola*; it is plain that they embrace both branches of the curve, and embrace them so closely that they form the boundaries of separation between all the transverse diameters of the curve, and all the conjugates to these:—every straight line drawn from O *within* the angle KON, or *within* the angle MOL, meets the curve; whilst every straight line drawn from O, *without* both these angles, that is within the angle MOK, or LON, proceeds onwards in both directions free of the curve.

All these inferences are fully justified by the equations (1) and (2) at p. 116: we have sufficiently seen that the straight line or lines (1) can meet the curve (2) only at a point infinitely remote, or for which $x = \infty$; but if (1) were replaced by the line $y = m'x$, in which m' is numerically *less* than $\frac{b}{a}$, there will always be a point (or rather *two* points) common to both the line and the curve, that is, a point for which the co-ordinates x, y, for the straight line, are the same as the co-ordinates for the curve; for it is obvious that m', being *less* than $\frac{b}{a}$, such a value may be given to x, that will render

$$\frac{b}{a}\sqrt{\left(1-\frac{a^2}{x^2}\right)} = m', \text{ or } \frac{b^2}{a^2}\left(1-\frac{a^2}{x^2}\right) = m'^2;$$

and therefore the y, corresponding to this value of x in the straight line, must be equal to the y corresponding to the same value of x in the curve. By solving the above equation, we find the expression for x^2 to be

$$x^2 = \frac{a^2}{1-\frac{a^2}{b^2}m'^2} \therefore x = \pm\sqrt{\frac{a^2}{1-\frac{a^2}{b^2}m'^2}},$$

so that the two values of x are always *real*, inasmuch as that m'^2 being *less* than $\frac{b^2}{a^2}$, the denominator of this fraction must always be *positive*. In the extreme case of $m'^2 = \frac{b^2}{a^2}$, the straight line, as we already know, becomes an asymptote; x *then* being infinite. If m'^2 were *greater* than $\frac{b^2}{a^2}$, the denominator would be negative, and x would be imaginary.

From a mere inspection of the curve, it is obvious that a line drawn parallel to an asymptote will cut the curve in one point, and in one point only: but we may prove this as follows.

The equation of the curve, and the equation of a parallel to the asymptote are

$$y^2 = \frac{b^2}{a^2}(x^2-a^2), \text{ and } y = \pm\frac{b}{a}x+h \quad . \quad . \quad . \ (1),$$

and therefore, at the point or points of intersection of the two, we must have

$$x^2-a^2 = \left(x\pm\frac{a}{b}h\right)^2 = x^2\pm2\frac{a}{b}hx+\left(\frac{a}{b}h\right)^2,$$

$$\therefore \ -a^2 = \pm\frac{a}{b}hx+\left(\frac{a}{b}h\right)^2,$$

in which equation x has but one value, and consequently y but one value, when the coefficient of inclination in (1) is $+\frac{b}{a}$, and also but one value when the coefficient is $-\frac{b}{a}$; hence *there* is but one point of intersection.

An asymptote may be regarded as the ultimate or limiting position of a tangent to the curve, the point of contact, in this ultimate position of the touching line, being infinitely remote. For if in the equation of the tangent at a point (x', y') of the hyperbola, that is, in the equation

$$a^2y'y - b^2x'x = -a^2b^2, \text{ we put } y' = \frac{b}{a}x' + \sqrt{\left(1 - \frac{a^2}{x'^2}\right)},$$

$$\text{it becomes } a^2y\frac{b}{a}x'\sqrt{\left(1 - \frac{a^2}{x'^2}\right)} - b^2x'x = -a^2b^2, \text{ or}$$

$$ay\sqrt{\left(1 - \frac{a^2}{x'^2}\right)} - bx = -\frac{a^2b}{x'};$$

and this, when $x' = \infty$, gives $y\sqrt{1} = \frac{b}{a}x$; or $y = \pm\frac{b}{a}x$; the equation of the asymptotes; so that when the point of contact, (x', y'), is infinitely remote, the tangent becomes an asymptote.

If the hyperbola be equilateral, that is, if $a = b$, then, (3), p. 117, tan KOB $= 1$ (See the diagram at p. 115), and tan MOB $= -1$; therefore KOB $= 45°$, and MOB $= 90° + 45°$: hence, in this case the asymptotes are at right angles to one another; on which account, the equilateral hyperbola is frequently called the *rectangular hyperbola.*

In what has preceded, respecting the asymptotes of an hyperbola, the principal diameters of the curve have been taken for axes of reference; but as the form of the equation of the curve, as also that of the equation of any straight line from the centre, is the same, whether the axes of reference are the principal diameters or any other pair of conjugates, it follows that $y = \pm\frac{b'}{a'}x$ will be the general form for the equations of the asymptotes.

If we put here $x = \pm a'$, we shall have for y, the values $y = +b$, and $y = -b$, consequently the tangents, Tt, T$'t'$, are each equal in length to the diameter $2b'$ or C$'$D$'$, which is the conjugate to $2a'$ or A$'$B$'$. It follows, therefore, that the vertices T, T$'$, t, t', of the inscribed parallelogram whose sides are parallel and equal to any pair of conjugate diameters A$'$B$'$, C$'$D$'$, are all on the asymptotes, and that each of the four sides is bisected by the conjugate to that diameter to which it is parallel. It has already been seen (p. 115) that each of these inscribed parallelograms is equal in surface to the rectangle of the principal diameters.

From what has just been said, we see how easy it is, when the position of the asymptotes OT, OT′ is given, to draw the conjugate to any assigned transverse diameter A′B′: all that is requisite is to draw a tangent T*t* to the curve at B′, and then a line C′D′, through the centre, parallel to T*t*, making OC′ = OD′ = B′T.

The Hyperbola referred to its Asymptotes as Axes.

The equation of the hyperbola, referred to its rectangular diameters as axes, is

$$a^2y^2-b^2x^2 = -a^2b^2,$$

and to transform the equation from these rectangular to a pair of oblique axes, we must substitute (p. 56)

$$x\cos\alpha+y\cos\alpha' \text{ for } x, \text{ and } x\sin\alpha+y\sin\alpha' \text{ for } y\,;$$

we thus get, as at p. 111,

$$\left.\begin{array}{l}a^2\sin^2\alpha'\\-b^2\cos^2\alpha'\end{array}\right|y^2+\left.\begin{array}{l}2a^2\sin\alpha\sin\alpha'\\-2b^2\cos\alpha\cos\alpha'\end{array}\right|xy+\left.\begin{array}{l}a^2\sin^2\alpha\\-b^2\cos^2\alpha\end{array}\right|x^2 = -a^2b^2 \quad . \quad . \quad . \quad (1).$$

Now let the new axis of x be that asymptote OK which makes the acute angle α with the principal semitransverse a, the other asymptote, OM, making the obtuse angle α', with that semitransverse, being the new axis of y: these angles are evidently so related that

$$\tan\alpha = \frac{b}{a}, \text{ and } \tan\alpha' = -\frac{b}{a}.$$

Moreover, since in a right-angled triangle, the sine of the base angle is equal to the perpendicular divided by the hypotenuse, and the cosine equal to the base divided by the hypotenuse, we have

$$\sin\alpha = \frac{b}{\sqrt{(a^2+b^2)}},\ \cos\alpha = \frac{a}{\sqrt{(a^2+b^2)}}\,;$$

$$\sin \alpha' = \frac{b}{\sqrt{(a^2+b^2)}}, \cos \alpha' = \frac{-a}{\sqrt{(a^2+b^2)}}$$

$$\therefore a^2 \sin^2 \alpha - b^2 \cos^2 \alpha = \frac{a^2b^2 - a^2b^2}{a^2+b^2} = 0$$

$$\text{and also } a^2 \sin^2 \alpha' - b^2 \cos^2 \alpha' = \frac{a^2b^2 - a^2b^2}{a^2+b^2} = 0.$$

Hence, when referred to the asymptotes, both x^2 and y^2 are absent from the equation of the curve; so that the term in xy is the only term involving the variables. The coefficient of this term, as appears from the above relations, is

$$2(a^2 \sin \alpha \sin \alpha' - b^2 \cos \alpha \cos \alpha') = 2(\frac{a^2b^2}{a^2+b^2} + \frac{a^2b^2}{a^2+b^2}),$$
$$= \frac{4a^2b^2}{a^2+b^2},$$

Consequently, the equation (1), under the above conditions, is simply

$$\frac{4a^2b^2}{a^2+b^2} xy = -a^2b^2, \text{ or } xy = -\frac{a^2+b^2}{4},$$

from which we see, since the right-hand member of the equation is necessarily negative, that every point in the curve, when referred to OK as axis of x, and OM as axis of y, must have its co-ordinates of opposite signs. And upon inspection of the curve we see that, with these axes of reference, every positive x has a negative y, and every negative x, a positive y. It is better therefore to interchange the axes, and not only so, but to regard ON as the axis of *positive* abscissas, OK being the axis of positive ordinates: the equation then becomes

$$xy = \frac{a^2+b^2}{4} \quad . \quad . \quad . \quad (2)$$

and the right-hand branch of the curve is embraced by the axis of positive abscissas, and the axis of positive ordinates. If we had at first taken ON for the new axis of x, and OK for the new axis of y, the angle BON $= \alpha$ would have been negative: it is equal in magnitude to the positive angle BOK $= \alpha'$: hence the formulas of transformation marked (2), at page 56, become in this case simply

for x, $(x+y)\cos a$, and for y, $(y-x)\sin a$
because $\cos(-a) = \cos(+a)$, and $\sin(-a)$
$= -\sin(+a)$; also $a' = -a$.

Substituting these values in the primitive equation

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

the transformed equation is

$$a^2(y-x)^2 \sin^2 a - b^2(x+y)^2 \cos^2 a = -a^2b^2.$$

But, p. 120,

$$\sin^2 a = \frac{b^2}{a^2+b^2}, \text{ and } \cos^2 a = \frac{a^2}{a^2+b^2},$$

$$\therefore \quad \frac{a^2b^2}{a^2+b^2}(y-x)^2 - \frac{a^2b^2}{a^2+b^2}(x+y)^2 = -a^2b^2,$$

$$\text{that is,} \quad -4\frac{a^2b^2}{a^2+b^2}xy = -a^2b^2;$$

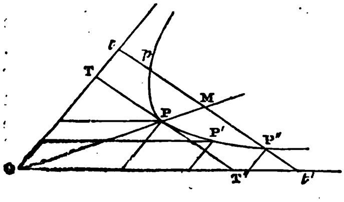
$$\therefore \quad xy = \frac{a^2+b^2}{4},$$

the equation of the curve when referred to its asymptotes, ON, OK, taken for the axes of x and y respectively.

It follows from this equation that the rectangle xy, of the co-ordinates of any point in the hyperbola referred to its asymptotes is a constant quantity, however remote the point (x, y) may be; and hence the following remarkable properties of the curve.

THEOREM 1.—All the parallelograms OP, OP′, OP″, &c. having O for a common vertex, the sides from O being on the asymptotes, and the opposite vertex of each parallelogram on the curve, are equal in area.

For the area of any parallelogram OP, or OP′, &c., is the rectangle of the two sides from O multiplied by the sine of the angle O formed by the asymptotes; that is, the area of each parallelogram, indifferently, is $xy \sin$ O; and we have just seen that



$$xy \sin \mathrm{O} = \frac{a^2+b^2}{4} \sin \mathrm{O}.$$

Hence the parallelograms are all equal in area.

This constant area is half the rectangle of the principal semidiameters. For $a^2+b^2 = \mathrm{AC}^2$, and $\mathrm{AC} = \mathrm{AD}$, because $\mathrm{OC} = \mathrm{OD}$. (See Diagram, p. 127).

$$\therefore a^2+b^2 = \mathrm{AC} \,.\, \mathrm{AD},\ \therefore \frac{a^2+b^2}{4} \sin \mathrm{O} = \frac{\mathrm{AC}}{2} \,.\, \frac{\mathrm{AD}}{2} \sin \mathrm{O};$$

and since the angle O = the angle CAD, this last expression is the area of half the triangle CAD, that is, of the triangle CAO, the area of which is also $\frac{1}{2}$ AO . OC, or $\frac{1}{2}\, a \,.\, b$.

THEOREM 2.—The triangle O T t (in the diagram at p. 115), cut off by any tangent to the curve, is always of constant area, and equal to the rectangle of the principal semidiameters.

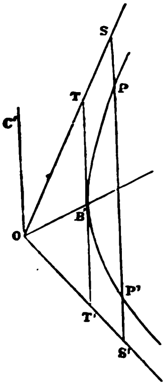
For the triangle O T t is equal to half the parallelogram T C′D′t, and this is half the parallelogram Ttt'T′, which parallelogram is of constant area so long as the adjacent sides of it TT′, Tt, included between the asymptotes, are parallel to a pair of conjugate diameters: hence the area of the triangle O T t is constant. Since the triangle is a fourth part of the inscribed parallelogram, the area of it is equal to a fourth part of the rectangle of the principal diameters, that is, to the rectangle of the principal semidiameters.

THEOREM 3.—If any chord pP″ be drawn, and produced both ways till it meet the asymptotes in t and t', the intercepted portions tp, t'P″, will always be equal.

Let pP″ be any chord produced to t, t', and TT′, a tangent parallel to it; then OM, through P, as it bisects TT′, bisects also not only tt', but likewise pP″ : hence the intercepts pt, P″t', between the curve and the asymptotes, are equal, in whatever direction the line tt' be drawn.

THEOREM 4.—If a straight line be drawn through any point P in the hyperbola, and terminating in the asymptotes, the rectangle of the two parts of it, between the point and the asymptotes, will be equal to the square of that semidiameter to which the line is parallel.

Let the curve be referred to the conjugate axes a', b', of which b' (or OC′) is parallel to the straight line SPS′ through P. Putting x for the abscissa of P, the ordinate of it is



$$= \frac{b'}{a'}\sqrt{(x^2-a'^2)},$$

whilst the ordinate of S is $\frac{b'}{a'}x$ (p. 119),

$$\therefore \text{SP} = \frac{b'}{a'}x - \frac{b'}{a'}\sqrt{(x^2-a'^2)}$$

and $$\text{PS}' = \frac{b'}{a'}x + \frac{b'}{a'}\sqrt{(x^2-a'^2)}$$

$$\therefore \text{SP} . \text{PS}' = b'^2.$$

NOTE.—Since, by Theorem 3, PS = P′S′, in whatever direction the line through P be drawn, it follows that when the asymptotes and but a single point P in the curve are given, as many other points in the curve as we please may be found; for if from *any* line SPS′, we cut off S′P′ = PS, P′ will be a point in the curve. And in like manner, from lines drawn through P′, may points in the upper arc be found. The intercepts are still equal when the chord unites *both* branches of the curve.

PROBLEM.—*To find the equation of the tangent to an hyperbola, when the curve is referred to the asymptotes as axes.*

As in the former cases, let us first consider the secant or line passing through two points of the curve (x', y'), and (x'', y'') : there must then exist the two equations

$$y-y' = \frac{y'-y''}{x'-x''}(x-x') \quad . \quad . \quad . \quad (1)$$

$$x'y' = x''y'', \text{ or } x'y'-x''y'' = 0 \quad . \quad . \quad . \quad (2)$$

If to this second equation we add the identity $-x'y''+x'y'' = 0$, it will take the form

$$x'(y-y'')+y''(x'-x'') = 0, \therefore \frac{y'-y''}{x'-x''} = -\frac{y''}{x'}.$$

Hence, by substitution, the equation (1) is

$$y-y' = -\frac{y''}{x'}(x-x'),$$

which when $x' = x''$, and consequently $y' = y''$, that is, when the two points coalesce, and the secant becomes a tangent, is

$$y-y' = -\frac{y'}{x'}(x-x'),$$

the equation of the tangent at the point (x', y'). Or, multiplying by x', and transposing,

$$x'y+y'x = 2x'y' = \frac{a^2+b^2}{2} \text{ (p. 122)} \quad . \; . \; . \quad (3)$$

The co-ordinates x, y, denote, of course, the co-ordinates of *any* point, indifferently, in the touching line: if we fix one of these points, denoting it by (x_1, y_1), then the equation

$$y_1x+x_1y = \frac{a^2+b^2}{2} \quad . \; . \; . \quad (4)$$

will be the equation of a straight line passing through the point of contact (x', y'), of the tangent drawn from the external point (x_1, y_1): for it is the equation of *some* straight line, and it is satisfied by the values x', y', the co-ordinates of the point of contact in (3): and if a second tangent be drawn from the same external point (x_1, y_1) to another point of contact (x'', y''), the first member of (3) would be $x''y+y''x$, and the equation (4) would also be satisfied for $x = x''$, and $y = y''$, the co-ordinates of the *second* point of contact: hence the straight line (4) passes through both, and is therefore *the chord of contact.* If then the straight line, of which the equation in reference to the asymptotes, is (4), be constructed, it will pass through the two points in which the two tangents from (x_1, y_1) will touch the curve.

To find the analytical expressions for the co-ordinates of these points of contact, we must solve the simultaneous equations

$$y_1x+x_1y = \frac{a^2+b^2}{2},$$

$$xy = \frac{a^2+b^2}{4}.$$

By putting k^2 for the second member of this last equation, we shall find the values of x and y, for the points of contact, to be

$$x = \frac{k^2 \pm k\sqrt{(k^2-x_1y_1)}}{y_1},$$

$$y = \frac{ky_1}{k \pm \sqrt{(k^2-x_1y_1)}}.$$

Should the point (x_1, y_1) be upon the curve itself, then would $x_1 y_1 = k^2$; and these expressions would become $x = x_1$, and $y = y_1$; as we know they ought to be, the *two* tangents being then but one continued straight line; the points (x_1, y_1), and (x', y'), are identical; and the equation (4) is the same as the equation (3).

If the co-ordinates of the external point have the same signs, and it were possible that the rectangle of these co-ordinates could exceed k^2, the co-ordinates of the points of contact would be imaginary; so that a tangent from the point (x_1, y_1) could not be drawn. But it is *not* possible, x_1, y_1 being both positive, or both negative, that $x_1 y_1$ should exceed k^2. For in order that the signs of the co-ordinates of the point may be the same, the point must be situated *within* the angle which includes one of the branches of the curve, as is obvious from inspection; and for a *tangent* to be drawn from it, it must be between the sides of this angle and the curve; if one of these co-ordinates (say y_1) be prolonged to meet the curve, and we put l for the prolongation, the co-ordinates of the point of meeting will be x_1, and $y_1 + l$; and since the rectangle of *these* is only k^2, the rectangle $x_1 y_1$ must always be less than k^2; of course l is to be taken with the same sign as y_1, it being the prolongation of y_1.

The foregoing expressions, therefore, for x and y, the co-ordinates of the two points of contact, must always be *real*, seeing that whenever the product $x_1 y_1$ is *plus*, it must be less than k^2; so that, let the external point be wherever it may, *two* tangents to the curve may always be drawn from it.

It may be noticed here, that, from the equation of the curve $xy = k^2$, we see that there can be but one ordinate to the same abscissa, and but one abscissa to the same ordinate; from which we infer, as indeed was otherwise proved at page 118, that a parallel to either of the asymptotes cuts the hyperbola in a single point only.

PROBLEM.—*To find the locus of the extremities of the conjugate diameters of an hyperbola.*

It was shown at page 112, that the square of any semi-conjugate diameter b' is

$$b'^2 = \frac{a^2b^2}{a^2 \sin^2 \alpha' - b^2 \cos^2 \alpha'} \quad . \quad . \quad . \quad (1)$$

where α' is the inclination of b' to a; so that putting x, y, for the rectangular co-ordinates of the extremity of b', the axes of reference being a, b, we shall have

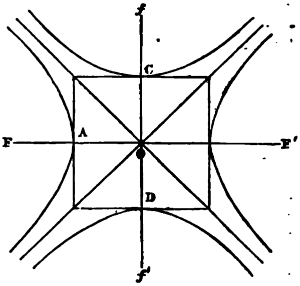
$$\sin \alpha' = \frac{y}{b'}, \quad \cos \alpha' = \frac{x}{b'}.$$

Substituting these values in the expression for b'^2 in (1), and then dividing by b'^2, there results

$$1 = \frac{a^2b^2}{a^2y^2 - b^2x^2}, \text{ or } b^2x^2 - a^2y^2 = -a^2b^2 \quad . \quad . \quad . \quad (3)$$

This is obviously the equation of an hyperbola whose principal semidiameters are b and a; b being the principal semitransverse, and a the principal semiconjugate.

The locus is therefore *an hyperbola*, in which all the transverse, or *real* diameters are the second, or *imaginary* diameters of the original hyperbola. It is situated in reference to this latter as in the annexed diagram, and the two, thus connected, are called *conjugate hyperbolas*: they have the same asymptotes in common.* For the tangents of the inclinations of the asymptotes of the primitive hyperbola to its principal transverse diameter, $2a$, are $\pm \frac{b}{a}$; and the tangents of the inclinations of the asymptotes of the conjugate hyperbola, to *its* principal transverse, $2b$, are $\pm \frac{a}{b}$: but these two diameters, $2a$, $2b$, are at right angles to one another, and $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$; so that what is *tangent*, in the



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* Of course the diagram is correspondent to that above when it is constructed in reference to oblique conjugates instead of rectangular conjugates; the inscribed parallelogram being then oblique.

one case, is *cotangent* in the other: hence, if α be the inclination of either one of the asymptotes of the primitive hyperbola to a, the complement of α will be the inclination of the corresponding asymptote, of the conjugate hyperbola, to b; consequently, these two asymptotes must coalesce; and thus the two hyperbolas have but one pair of asymptotes.

It is obvious, too, that a straight line through the extremity C of any second diameter of one hyperbola, and parallel to the transverse to that diameter, is a tangent to the conjugate hyperbola at the point C; and that the inscribed parallelogram adverted to at page 115, always has its four sides in contact with the four branches of the conjugate hyperbolas, each side being a tangent; and that the four vertices of the parallelogram are upon the asymptotes.

We shall merely further observe, that the four foci are all equidistant from the common centre O, the expression for the square of this distance, whichever hyperbola be taken, being $\frac{a^2+b^2}{a^2}$.

Examples for Exercise.

1. If a normal be drawn through any point of an equilateral hyperbola, the part of it intercepted between the principal axes will be bisected at the point; but if the hyperbola be not equilateral, the parts are to one another as the squares of the principal diameters.

2. Prove that the locus of the extremities of perpendiculars from the foci of an hyperbola upon tangents to the curve is a circle.

3. If tangents be drawn at the extremities of any focal chord, prove that they must intersect on the *directrix*, or the line whose equation is $x = \frac{a}{e}$.

4. Prove that a straight line drawn from the focus to the intersection of the tangents in last example is perpendicular to the focal chord.

5. If two chords in an equilateral hyperbola intersect one another, prove that the rectangle of the segments of the one will be equal to the rectangle of the segments of the other.

6. If at the extremities of any transverse diameter A′B′, tangents be drawn, and these meet a third tangent to the curve in t and t', prove that the rectangle $A't' \cdot B't = b'^2$.

7. Prove that in an equilateral hyperbola a pair of focal chords parallel to a pair of conjugate diameters are equal.

8. If an ellipse and an hyperbola have the same centre and the same axes, $2a$, $2b$, prove that the equal conjugate diameters of the ellipse will be upon the asymptotes of the hyperbola.

9. Two tangents are drawn to an hyperbola, and the points where they cross the asymptotes are joined: prove that these joining lines are parallel.

10. The angle (θ) between the asymptotes, and a point (x', y'), in the hyperbola, being given, to determine the principal semidiameters of the curve.

Ans. $a = 2\cos\frac{1}{2}\theta\ \sqrt{x'y'}$, $b = 2\sin\frac{1}{2}\theta\ \sqrt{x'y'}$.

SECTION X.—ON THE LOCI OF EQUATIONS OF THE SECOND DEGREE.

The equations of the curves discussed in the three preceding sections have each been deduced from the condition, or law, which regulates the generation or description of the curve; and we have seen that, in each case, the equation which, in an algebraical form, embodies the distinctive peculiarities of the curve, is of the second degree. On this account the *conic sections*, as these curves are called, are said to be *curves of the second order*. But although, as we have thus seen, the conic sections may each be analytically represented by an equation of the second degree, it remains still to be ascertained whether, conversely, an equation of the second degree, with two variables, must necessarily represent a conic section, whenever it represents any curve at all. This qualification is necessary, because an equation of the second degree may be but the product of two equations of the first degree; and may therefore be the algebraical representation of two straight lines merely, or indeed of only a single straight line; for the component equations of the first degree may be identical, each representing one and the same straight line.

When, therefore, it is stated that every equation of the second degree involving two variables, x and y, must of necessity be the analytical representation of one or other of the three curves already discussed, the statement is to be understood as implying that no *curve* which is not one of the three conic sections can possibly be the locus of the point (x, y); and that when the equation instead of representing one of these curves characterises only a straight line or a pair of straight lines, the circumstance can arise only from some *special* relation having place among the coefficients of the equation.

It is now our business to examine into this matter; and to prove that when any equation of the second degree, with two variables, is not in the peculiar predicament adverted to above, that is, when its constitution is such that no *straight* line is represented by it, it must be the equation of a conic section, and can never be the algebraical representation of any other curve whatever.

The proposed investigation will consist of three distinct divisions: it will be proved—

First, that every equation which corresponds in form to one or other of the equations known to characterise a conic section can be the algebraical representation of no other curve except that one.

Secondly, that of whatever form the proposed equation may be, it may always be transformed into another agreeing with that of a conic section; and—

Thirdly, it will be proved that in every equation, the particular one of the three curves represented, may always be ascertained by examining the relations among the coefficients of that equation.

1st. *To determine the locus of the equation when of the form*

$$\mathrm{M}y^2+\mathrm{N}x^2=\mathrm{P} \quad . \quad . \quad . \quad (1), \text{ or } y^2=\mathrm{Q}x \quad . \quad . \quad . \quad (2).$$

The first of these agrees in form with the equation of the ellipse or with that of an hyperbola: the second equation corresponds with that of a parabola; and it is now to be shown that (1) necessarily represents an ellipse or an hyperbola, whenever it represents any curve at all; and that (2) applies to no other curve but a parabola.

In the equation (1) put a^2 for $\frac{P}{N}$, and b^2 for $\frac{P}{M}$; then we shall have $N = \frac{P}{a^2}$, $M = \frac{P}{b^2}$, and for the equation (1),

$$\frac{P}{b^2}y^2 + \frac{P}{a^2}x^2 = P, \text{ or } \frac{1}{b^2}y^2 + \frac{1}{a^2}x^2 = 1,$$

which we know to be the equation of an ellipse when a^2, and b^2, are both positive, and the equation of an hyperbola when one of these quantities is positive and the other negative. If it be a^2 that is negative, the hyperbola, which in that case is represented by $\frac{1}{a^2}x^2 - \frac{1}{b^2}y^2 = -1$, will be the conjugate to the hyperbola represented by $\frac{1}{b^2}y^2 - \frac{1}{a^2}x^2 = -1$, the equation when b^2 is negative.

If, however, P alone is negative, M and N being positive, then the equation has no geometrical signification; for since, in this case,

$$y = \sqrt{-\left(\frac{N}{M}x^2 + \frac{P}{M}\right)},$$

we see that for every real value of the abscissa x, the ordinate y must be imaginary; so that no real locus exists. But if P is zero, M and N being still positive, the equation will represent merely a point, namely, the point (0, 0), the origin of the co-ordinate axes: this point, and the circle which (1) denotes when $M = N$, and P positive, are called the *varieties* of the ellipse.

If when P is zero one of the two coefficients, N for instance, is negative, the other coefficient M being positive, the equation gives

$$y = \pm\sqrt{\frac{N}{M}}\,x, \text{ that is, } y = \pm\frac{b}{a}x,$$

which we know represents the asymptotes of the hyperbola

$$\frac{1}{b^2}y^2 - \frac{1}{a^2}x^2 = -1.$$

Hence under these conditions as to the coefficients in equation (1), that is, when P = 0, and either M or N negative, and the other positive, the curve which, for any significant value of P, would be an hyperbola, degenerates, in this extreme case of P zero, into a system of two straight lines passing through the origin; the equations of these straight lines being

$$y\sqrt{\text{M}}+x\sqrt{\text{N}} = 0, \text{ and } y\sqrt{\text{M}}-x\sqrt{\text{N}} = 0.$$

This system of two intersecting straight lines is a *variety* of the hyperbola.

We may notice here that an infinite number of different hyperbolas may all have the same asymptotes; for the equation

$$a^2y^2-b^2x^2 = -a^2b^2, \text{ being that of an hyperbola,}$$

so also is $(na)^2y^2-(nb)^2x^2 = -(na)^2(nb)^2$; the principal semidiameters of the former being a, b, and the principal semidiameters of the latter na, nb. In the one curve the equations of the asymptotes are

$$y = \frac{b}{a}x, \; y = -\frac{b}{a}x:$$

$$\text{in the other, } y = \frac{nb}{na}x, \; y = -\frac{nb}{na}x;$$

which pairs of equations are identical. As n diminishes from $n = 1$ towards $n = 0$, the semidiameters na, nb, approach nearer and nearer to zero; and the curve ultimately degenerates into the two straight lines which, up to this extreme limit, had been the pair of asymptotes common to all the varying hyperbolas; just as the varying ellipse degenerates into a circle, and then this circle into a point. And hence the propriety of calling the circle and point *varieties* of the ellipse; and two intersecting straight lines a variety of the hyperbola.

As to the equation (2), we at once conclude, from its form, that it represents a parabola; the constant Q being its principal parameter. Even if Q were negative, the conclusion would be the same; only in this case, the equation would imply that the curve is situated on the *negative* side of the axis of y; the abscissa of each point in it being negative.

The smaller Q becomes, the closer do the portions of the curve—the one portion above, and the other portion below, the axis of x—approach one another; till at length, when Q vanishes, the two portions coalesce and become confounded with the axis of x. This single straight line is therefore a *variety* of the parabola.

There is another variety of this curve: suppose the origin of the axes to be moved forwards in the positive direction, the parabola itself remaining fixed; or, which is the same thing as respects the relative positions of the axes and the curve, suppose the vertex of the curve to be moved along the axis of x, in the negative direction, to a distance k from its original position: the equation of the curve will then be

$$y^2 = Q(x+k) = Qx+Qk,$$

which, whatever be Q and k, characterises a parabola, except the relation between these arbitrary constants be such that the curve degenerates into one of its varieties.

Let the distance k of the vertex of the parabola be taken greater and greater, and Q smaller and smaller, but always so that the rectangle Qk may remain constant. In order to this, it will only be necessary that any two values k, k', be related to the contemporaneous values Q, Q', so that

$$k : k' :: Q' : Q, \text{ for then } k'Q' = kQ = K,$$

K will thus be a constant quantity, and the equation of the parabola will be $y^2 = Qx+K$, which, when Q has continuously diminished down to zero (and consequently k has become infinite) merges in the equation

$$y^2 = K, \therefore y = \sqrt{K}, \text{ or } y = +\sqrt{K}, \text{ and } y = -\sqrt{K};$$

that is, the parabola degenerates into a pair of straight lines, parallel to one another, and equidistant from the axis of x. And these two straight lines unite in a single line, the axis of x, when K is zero.

We thus see that the equation $y^2 = Qx$ always represents either a parabola or else one of the varieties of a parabola,—a single straight line coincident with the axis of x, or a pair of straight lines parallel to, and equidistant from, the axis of x.

2nd. *To determine the locus when the equation is unrestricted in form.*

We now proceed to prove that the locus of every equation of the second degree, containing two variables, x and y, can be no other curve than one of the three curves already considered. The truth of this will be established provided it be shown that the general equation

$$\text{A}y^2+\text{B}xy+\text{C}x^2+\text{D}y+\text{E}x+\text{F}=0 \quad . \quad . \quad . \quad (1)$$

may always be transformed into another either of the form

$$\text{M}y^2+\text{N}x^2=\text{P}, \text{ or of the form } y^2=\text{Q}x,$$

by merely changing the axes to which the locus (1) is referred.

The completeness of the proposed proof will not be impaired if we regard the axes to which (1) is referred as *rectangular* axes; since if they were originally oblique they might have been replaced by rectangular axes, and it is impossible that the resulting transformed equation could contain any terms in which the general equation (1) is deficient.

I. *To transform the equation* (1) *into another in which the product of the variables shall be absent.*

For x and y in the equation (1) substitute, respectively,

$$x \cos a - y \sin a, \text{ and } x \sin a + y \cos a \quad . \quad . \quad . \quad (2)$$

by means of which substitutions we pass from one system of rectangular axes to another system also rectangular, the origin remaining undisturbed. The coefficient of xy, in the resulting transformed equation, will be found to be

$$2\text{A} \sin a \cos a + \text{B} \cos^2 a - \text{B} \sin^2 a - 2\text{C} \sin a \cos a.$$

Consequently if the angle a, through which the original rectangular axes are to be turned, be so determined that this coefficient may become zero, xy will disappear from the equation of the locus when referred to the new axes. The equation for determining the proper value of a is therefore

$$2(A-C)\sin\alpha\cos\alpha+B(\cos^2\alpha-\sin^2\alpha)=0,$$
$$\text{or } (A-C)\sin 2\alpha+B\cos 2\alpha=0,$$

which implies the condition $\tan 2\alpha=\frac{B}{C-A}$; so that if in the formulas (2) we give to the angle α a value such that the tangent of the double of it may be the number $\frac{B}{C-A}$, no term containing xy can appear in the transformed equation. Hence by turning the rectangular axes through this angle α, the equation of the locus will be of the form

$$My^2+Nx^2+Ry+Sx+F=0 \quad . \quad . \quad . \quad (3).$$

II. *To transform the equation* (3) *into another in which the first power of the variables shall be absent.*

This transformation is effected by simply shifting the origin of the axes, without disturbing their directions. In order that the origin may be removed from the point (0, 0) to the point (a, b), we have only to substitute $x+a$ for x, and $y+b$ for y; and with these substitutions the equation (3) is transformed into

$$My^2+Nx^2+(2Mb+R)y+(2Na+S)x-P=0,$$

where $-P$ is put for the aggregate of the quantities independent of the variables.

Now in order that the terms containing y and x, in this equation may disappear, we have only so to determine the arbitrary constants a, b, that the two conditions following may be satisfied, namely, the conditions

$$2Mb+R=0, \text{ and } 2Na+S=0,$$
$$\therefore\ b=-\frac{R}{M}, \text{ and } a=-\frac{S}{N}.$$

It follows, therefore, that by fixing the values of the three disposable constants α, a, and b, so that they may conform to the three conditions,

$$\tan 2\alpha=\frac{B}{C-A},\ a=-\frac{S}{2N},\ b=-\frac{R}{2M},$$

the equation (1) becomes transformed into the equation

$$My^2+Nx^2 = P \quad . \quad . \quad . \quad (4)$$

provided, that is, that neither My^2 nor Nx^2 has disappeared from the transformed equation (3).

Suppose, however, that one of these terms is absent from (3), as for instance the term Nx^2, then N would, of course, be absent from the subsequent transformed equation, and the coefficient of x in that equation would be simply S, the same as in (3) itself; and this, being a definite constant, we are forbidden to equate to zero. If therefore the term Sx be not absent from (3) as well as the term Nx^2, it cannot be removed from the transformed equation; no disposable constant (as the constant a) can in this case enter the coefficient of x; the disposable constant a does, however, enter the absolute term $(-P)$ of the transformed equation; for this term is

$$-P = Mb^2+Rb+Sa+F, \text{ which, equated to zero, gives}$$

$$a = -\frac{Mb^2+Rb+F}{S};$$

and with this value of a, and the value determined above for b, namely, $b = -\frac{R}{M}$, the resulting transformed equation is

$$My^2+Sx = 0; \text{ or putting } Q \text{ for } -\frac{S}{M},\ y^2 = Qx.$$

In like manner if My^2 had been absent from (3) instead of Nx^2, the transformed equation would have been

$$Nx^2+Ry = 0, \text{ or putting } Q' \text{ for } -\frac{R}{N},\ y^2 = Q'x.$$

We cannot assume that M and N are *both* of them zero in the transformed equation (3) because then the equation would not be one of the second degree. But if the terms containing the first and second powers of the same variable, the terms Sx and Nx^2, for instance, be absent from (3), then the equation will be an equation of the second degree with only one variable, namely, the equation

$$My^2+Ry+F = 0,$$

$$\text{which gives } y = -\frac{R}{M} \pm \frac{1}{2M}\sqrt{(R^2-4FM)},$$

showing that in these circumstances the locus will be a pair of parallel straight lines, which, however, are *real* lines, only so long as R^2 exceeds 4FM: if such be not the case the lines are imaginary.

This *variety* of the general equation, arising from the conditions that $\text{N} = 0$, and $\text{S} = 0$, in the transformed equation (3), is a variety of the parabola, because when N is absent from (3) that equation is of the same form as the equation of the parabola referred to rectangular axes, parallel to the principal axes of the curve, as is obvious from the formulas of transformation (1) at page 56.

It thus appears that every equation of the second degree involving two variables characterises either an ellipse, an hyperbola, or a parabola, or else a *variety* of one or other of these curves.

It remains for us now to show how to ascertain what particular conic section it is that any proposed equation represents.

3rd.—*Discussion of the general equation.*

$$\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0.$$

Solving this equation first for y and then for x, we get

$$y = -\frac{\text{B}x + \text{D}}{2\text{A}} \pm$$

$$\frac{1}{2\text{A}}\sqrt{\{(\text{B}^2 - 4\text{AC})x^2 + 2(\text{BD} - 2\text{AE})x + (\text{D}^2 - 4\text{AF})\}} \quad . \quad (1)$$

$$x = -\frac{\text{B}y + \text{E}}{2\text{C}} \pm$$

$$\frac{1}{2\text{C}}\sqrt{\{(\text{B}^2 - 4\text{AC})y^2 + 2(\text{BE} - 2\text{CD})y + (\text{E}^2 - 4\text{CF})\}} \quad . \quad (2)$$

Let us now enquire what conditions must be satisfied by the equation, in order that the expression under the radical sign in (1) must necessarily be *positive*, and consequently the values of x and y always possible. The sign of the expression referred to will not be altered, though we divide by x^2; since x^2 is *positive*, whether x be plus or minus: hence the sign of the irrational part of (1) will be the same as the sign of the expression

$$(\text{B}^2 - 4\text{AC}) + 2(\text{BD} - 2\text{AE})\frac{1}{x} + (\text{D}^2 - 4\text{AF})\frac{1}{x^2},$$

whatever value for x be put in each of them. Now a value may be given to x such that for it, and for all *greater* values, the sign of the entire expression just written may be the same as the sign of the first term of it; seeing that for the extreme value $x = \infty$, the second and third terms vanish altogether:—there *must* be a finite value for x so great that the last two terms, taken together, shall be numerically *less* than the first term; and therefore less for every value of x still greater. In other words, there exists some value for x such that for that value, and for all values positive and negative, numerically greater, the expression (1), for y, will be *real* if B^2-4AC be *positive*, and *imaginary* if B^2-4AC be negative.

Now it has already been shown that the only curve which the equation (1) can represent, must be either an ellipse, an hyperbola, or a parabola; and the only one of these of which—after a certain pair of values, positive and negative, for x—the ordinates are all *real*, however great $\pm x$ may be, is the *Hyperbola*: and the only one of the three curves in which, after a certain pair of values, one value positive and the other negative, for x, the ordinates are all *imaginary*, is the *Ellipse*. We may therefore safely conclude that—

If (B^2-4AC) be *negative*, the locus is an Ellipse
If (B^2-4AC) be *positive* „ „ Hyperbola
$\therefore$ If (B^2-4AC) be 0 „ „ Parabola;

this latter criterion for the parabola being a necessary consequence of the two former criteria. If, however, in conjunction with this third condition, there be the additional restriction $BD-2AE=0$, then the equation will represent—not a *curve*, but the two parallel straight lines whose equations are

$$y = -\frac{Bx+D}{2A} + \sqrt{(D^2-4AF)}$$

$$\text{and } y = -\frac{Bx+D}{2A} - \sqrt{(D^2-4AF)}.$$

That these lines are *parallel*, we know; because the coefficient of inclination is the same in both. As already noticed, these lines are called a *variety* of the parabola: the two lines coalesce and become one line, if $D^2 = 4AF$; as is obvious.

If the roots of the expression under the radical sign in (1) be *equal*, then, calling each of the equal roots r, the equation (1) will represent two straight lines, whenever the criterion for the hyperbola has place: the equations of these lines being

$$y = -\frac{Bx+D}{2A} + \frac{\sqrt{(B^2-4AC)}}{2A}(x-r_1)$$

$$\text{and } y = -\frac{Bx+D}{2A} - \frac{\sqrt{(B^2-4AC)}}{2A}(x-r_1),$$

and these, we know, must be *intersecting* straight lines, because the coefficient of x in the one is unequal to the coefficient of x in the other. This pair of intersecting straight lines is therefore a *variety* of the hyperbola.

From what has now been proved it follows that, whatever be the form of any equation of the second degree, involving two variables, the particular curve represented by it may always be ascertained by subjecting the given coefficients to the tests at page 138; so that we need never be in doubt, in any special case, whether the locus is an ellipse, an hyperbola, or a parabola. There is, however, one form of the equation which deserves distinct notice; we know from the criteria just referred to, that if either Ay^2, or Cx^2, be absent from the equation provided Bxy be present, the locus is necessarily an hyperbola, and that such is also the case if both Ay^2 and Cx^2 be absent, the term Bxy being present. Under these last circumstances the equation is

$$Bxy + Dy + Ex + F = 0, \therefore y = -\frac{Ex+F}{Bx+D};$$

which shows that when the equation is of this form, the axes, to which the hyperbola represented by it is referred, must be parallel to the asymptotes, since y is real for *every* real value of x; and that, moreover, for every such value of x, there corresponds but *one* value for y.

We shall only add further that the equations

$$Y = -\frac{BX+D}{2A} \;\ldots\; (3) \text{ and } X = -\frac{BY+E}{2C} \;\ldots\; (4)$$

which would be given by suppressing the irrational part in each of the expressions in (1) and (2), always represent two *diameters* of the locus: for whatever value be given to x in the irrational part of equation (1), if we call that value k, the y of the curve will be

$$y = -\frac{\mathrm{B}x+\mathrm{D}}{2\mathrm{A}} \pm k \quad . \; . \; . \; (5)$$

so that if from the point (X, Y) in the line (3), we draw the line k in the positive direction, parallel to the axis of y, and also the equal line $-k$, in the opposite direction, the extremities of k, $-k$, will be two points in the curve, the ordinates of which are the two values of y marked (5) above: consequently (3), thus bisecting every chord of the curve parallel to the axis of y, is the equation of a diameter; and in like manner it may be shown that (4) is the equation of a diameter bisecting the chords parallel to the axis of x. Hence, the *centre* of the curve, when it has a centre, must be at the intersection of the two lines (3), and (4); which point (x', y') of intersection, is found to be

$$x' = \frac{2\mathrm{AE}-\mathrm{BD}}{\mathrm{B}^2-4\mathrm{AC}},\; y' = \frac{2\mathrm{CD}-\mathrm{BE}}{\mathrm{B}^2-4\mathrm{AC}};$$

and each of these co-ordinates, when $\mathrm{B}^2-4\mathrm{AC}$ is zero, that is, when the curve is a parabola (p. 138), becomes infinite, as it ought to do.

From the general equation to determine the equations of the asymptotes.

By actually extracting the square root of the expression under the radical sign, in the general value of the ordinate in equation (1), we find for y the expression

$$y = -\frac{\mathrm{B}x+\mathrm{D}}{2\mathrm{A}} \pm \left\{ \frac{\sqrt{(\mathrm{B}^2-4\mathrm{AC})}}{2\mathrm{A}}\,x + \frac{\mathrm{BD}-2\mathrm{AE}}{2\mathrm{A}\sqrt{(\mathrm{B}^2-4\mathrm{AC})}} + \frac{k}{x} + \frac{k'}{x^2} + \;.\;.\;.\; \right\},$$

k, k', &c., being constants.

As x increases, it is plain that the term $\frac{k}{x}$, and all the terms which follow it will diminish, whilst those in x that precede it will increase; and to these leading terms the *expression* will ultimately be reduced when x becomes infinite. Consequently the curve continually approaches closer and closer to the two straight lines represented by the equations

$$y = -\frac{\text{B}x+\text{D}}{2\text{A}} + \frac{\sqrt{(\text{B}^2-4\text{AC})}}{2\text{A}}\left(x+\frac{\text{BD}-2\text{AE}}{\text{B}^2-4\text{AC}}\right)$$

$$\text{and } y = -\frac{\text{B}x+\text{D}}{2\text{A}} - \frac{\sqrt{(\text{B}^2-4\text{AC})}}{2\text{A}}\left(x+\frac{\text{BD}-2\text{AE}}{\text{B}^2-4\text{AC}}\right).$$

These two lines are therefore the asymptotes to the curve: they are identical with those of which the equations are given at p. 139, as they ought to be since the pair of lines designated there as a variety of the hyperbola, are the asymptotes here. That they are identical pairs will be seen by substituting in the former pair of equations, the proper expression for r_1, one of the two *equal roots* of the equation furnished by putting the expression under the radical sign in (1) equal to zero. This equation, when divided by the coefficient of x^2 gives for r_1, that is, for half the coefficient of x taken with changed sign, the value

$$r_1 = -\frac{\text{BD}-2\text{AE}}{\text{B}^2-4\text{AC}}$$

and it is at once seen that, with this value of r_1, the two equations at p. 139 are identical with the two equations of the asymptotes here determined.

As the centre of the hyperbola is the point where the asymptotes intersect, and at this point of intersection the values of y in the two foregoing equations must be identical, their *difference* must be zero: subtracting therefore the second equation from the first, and equating the result to zero, we find, for the abscissa X of the centre of the curve, the expression

$$\text{X} = \frac{2\text{AE}-\text{BD}}{\text{B}^2-4\text{AC}}.$$

And since half the *sum* of the values of y must be the ordinate Y of the centre, we have

$$\text{Y} = -\frac{\text{BX}+\text{D}}{2\text{A}} = \frac{2\text{CD}-\text{BE}}{\text{B}^2-4\text{AC}}.$$

These, indeed, are the general expressions for the coordinates of the centre in both the *central curves*, as the ellipse and hyperbola are called: this we learn from the equations of the pair of diameters at p. 139; which, solved as two simultaneous equations, give the foregoing values for X and Y.

We here terminate the proposed general discussion, and shall conclude the present Rudimentary Treatise by adding a few examples on the determination of Loci in particular cases.

And it may be as well to state that, in order to this determination, the student ought to proceed independently of all reference to the *criteria* at p. 138. It will be an easy thing for him to remember these simple tests for distinguishing the equation of any one of the three curves from the equation of either of the other two: he has only to call to mind the tests which he would employ, in a quadratic expression, in common algebra, in order to discover whether that expression is a complete square, or a quantity in *defect* of a complete square, or a quantity in *excess* of a complete square: this test being applied only to the three *leading* coefficients of the *arranged* equation—the terms of it which follow being wholly disregarded. It cannot but be regarded by him as an interesting fact that, by a mere inspection of these three leading terms, he should be enabled to declare, with certainty, whether the curve which the equation symbolises is a parabola, an ellipse, or an hyperbola. These names of the curves are significant of the above-mentioned discriminating peculiarities: parabola implying *equality*, ellipse, *defect*, and hyperbola, *excess*.

Problems on the determination of Loci.

1. It is required to determine the curve of which the equation, in reference to assigned axes,* is

$$y^2-2xy-3x^2-2y+7x-1=0.$$

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* In all such problems as these, the axes of reference are regarded as given in position: the situation, or inclination, of the axes has nothing to do with the general character of the curve; but the particular curve itself, in order that it may be actually constructed, or even that the situation of a single point in it may be determined, must of course be referred, by the equation of it, to a definite pair of axes.

Since in this equation B^2-4AC is *positive*, we at once conclude that the curve is an hyperbola. Let us first find the equations of its asymptotes, since when these lines are constructed, and a single point only in the curve determined, we can readily discover as many more points in it as we please (p. 124), and can thus trace out the particular hyperbola.

The equations of the asymptotes are given in a general form at p. 141; but in particular cases it is usually preferable to proceed independently of the general formulas: we shall here therefore deduce the equations of the asymptotes directly from the proposed equation. Solving the equation for y, we get $y = x+1 \pm \sqrt{(4x^2-5x+2)}$ or actually extracting the square root,

$$y = x+1 \pm (2x-\tfrac{5}{4}+\frac{k}{x}+\frac{k'}{x^2}+\quad .\ .\ .\ \ ).$$

Hence (p. 140), we conclude that the equations of the asymptotes are

$$Y = x+1+2x-\tfrac{5}{4}, \text{ and } Y = x+1-2x+\tfrac{5}{4}.$$

Putting $x = 0$ in each of these we find $Y = -\frac{1}{4}$, and $Y = 2\frac{1}{4}$, for the ordinates of the points where the two lines cross the axis of y. In like manner, putting $Y = 0$, the points where the same lines cross the axis of x are found to be $x = \frac{1}{12}$, and $x = 2\frac{1}{4}$: the asymptotes may therefore be easily drawn.

It remains now to determine a point in the curve. For this purpose put $x = 0$, in the proposed equation; and there results for the ordinates of the two points where the curve crosses the axis of y, the values $y = 1+\sqrt{2}$, and $y = 1-\sqrt{2}$.

Two points in the curve are thus ascertained, and by aid of these we may find as many more as we please, and may thus be enabled to trace out the curve.

If in any case it should happen that the curve does not cross either of the axes of reference, then after having constructed the asymptotes, a point in the curve is to be determined by assuming some convenient value for x, and then from the given equation determining the corresponding value of y: by help of this point, and the two asymptotes, other points in the curve—as many as we choose—may be discovered, as already explained (p. 124).

2. It is required to determine the curve of which the equation is

$$y^2-2xy+3x^2+2y-4x-3=0.$$

Here B—4AC is negative: the curve must therefore be an ellipse. We shall determine its limits.

Writing the equation in the forms $y^2-2(x-1)y=-3x^2+4x+3$, and $3x^2-2(y+2)x=-y^2-2y+3$, and solving it first for y, and then for x, we have

$$y=x-1\pm\sqrt{(-2x^2+2x+4)} \quad . \quad . \quad . \quad (1)$$

$$x=\tfrac{1}{3}(y+2)\pm\tfrac{1}{3}\sqrt{(-2y^2-2y+13)} \quad . \quad . \quad . \quad (2)$$

And by equating the expression under the radical sign in (1) to zero, we have $x^2-x-2=0$, $\therefore x=\frac{1\pm3}{2}=2$, or -1.

The curve, therefore, is included between two parallels to the axis of y, of which the abscissa of the one is 2, and the abscissa of the other -1.

In like manner, equating the expression under the radical sign in (2) to zero, we have

$$2y^2+2y-13=0,$$

$$\therefore\ y=\frac{-1\pm\sqrt{27}}{2}=\frac{-1+\sqrt{27}}{2},\ \text{or}\ \frac{-1-\sqrt{27}}{2}.$$

The curve, therefore, is also included between parallels to the axis of x, of which the ordinate of the one is $\frac{-1+\sqrt{27}}{2}$ and the ordinate of the other $\frac{-1-\sqrt{27}}{2}$. The curve is therefore circumscribed by a parallelogram which may be constructed, and the points of contact found.

3. To determine the curve of which the equation is $xy-2y+x-1=0$.

The curve is an hyperbola (see p. 139), the axes of co-ordinates being parallel to the asymptotes. The expression for y is $y=\frac{1-x}{x-2}$, which becomes infinite only when

$x = 2$: hence, if this distance be measured from the origin on the axis of x, the line through its extremity, parallel to the axis of y, will be one of the asymptotes. In like manner the expression for x, viz. $x = \frac{2y+1}{y+1}$, becomes infinite only when $y = -1$; and if this distance be marked off from the origin on the axis of y, the line through its extremity, parallel to the axis of x, will be the other asymptote.

To find a point in the curve, put $x = 0$, in the given equation, and there results for the corresponding y, the value $y = -\frac{1}{2}$; therefore the point $(0, -\frac{1}{2})$ is a point in the curve; and by means of this and the asymptotes as many points as we please may be determined (Note, p. 124).

4. Given the base of a triangle, and the sum of the tangents of the angles at the base, to determine the locus of the vertex of the triangle.

Let the middle point of the base be taken for the origin of the rectangular co-ordinates, the axes being the base itself, for the axis of x, and the perpendicular to it, for the axis of y; and let (x, y) denote any point, indifferently, in the locus of the vertex. Then putting b for half the base, and s for the given sum of the tangents, we shall have, for the angles A and B, at extremities of the base $2b$,

$$\tan \text{A} = \frac{y}{b+x}, \tan \text{B} = \frac{y}{b-x};$$

and for the sum (s) of these tangents,

$$s = \frac{2by}{b^2-x^2}. \therefore sx^2+2by-sb^2 = 0,$$

$$\text{or } sx^2+2b\left(y-\frac{sb}{2}\right) = 0 \quad . \quad . \quad . \quad (1),$$

the equation of the curve. The locus of the vertex of the triangle is therefore a parabola.

If the origin of the co-ordinates be removed to a point in the axis of y, of which the ordinate is $\frac{sb}{2}$, that is, if we substitute $y+\frac{sb}{2}$ for y in equation (1), we shall have

$$sx+2by = 0, \therefore x^2 = -\frac{2b}{s}y \quad . \quad . \quad . \quad (2),$$

so that the vertex of the parabola is the vertex of the triangle when its position is on the axis of y; and it is this point which is the new origin of the axis. If in the equation of the locus (2) we put $-\frac{sb}{2}$ for y, the resulting values of x will be $x = +b$, and $x = -b$, showing that the parabola passes through the extremities of the fixed base of the triangle.

We subjoin a few unsolved problems of this kind.

5. Determine the character of the curve of which the equation is

$$y^2 - 4xy + 4x^2 + 2y - 7x - 1 = 0,$$

and find at what points it crosses the axes of co-ordinates?

Ans. The curve is a parabola: it intersects the axis of x in the points $x = \frac{1}{8}(7 \pm \sqrt{65})$, and the axis of y, in the points $y = -1 + \sqrt{2}$.

6. What are the equations of the asymptotes of the hyperbola $xy - 2y + x - 1 = 0$, and at what point does the curve intersect the axis of ordinates?

Ans. Equations of asymptotes, $x = 2$, and $y = -1$. Point, $y = -\frac{1}{2}$.

7. What is the character of the curve represented by the equation

$$x^2 + mxy - y^2 - n = 0,$$

the axes of reference being rectangular?

Ans. An equilateral hyperbola.

8. Given the base and altitude of a plane triangle, to find the locus of the intersections of the perpendiculars from the vertices to the opposite sides of the triangle.

Ans. A parabola.

9. Given the base of a plane triangle and the difference of the tangents of the angles at the base to find the locus of the vertex? *Ans.* An hyperbola.

10. The extremities of a straight line of given length move along the sides of a given angle: it is required to find the locus of a fixed point P in the line?

Ans. The locus of P is an ellipse.

11. Given the base and the sum of the sides of a plane triangle, to find the locus of the point of intersection of the lines from the vertices bisecting the opposite sides?

Ans. An ellipse.

12. Tangents to a parabola make angles with the principal diameter, the product of whose tangents is given: required the locus of their points of intersection?

Ans. A straight line.

13. A perpendicular from any point in the diameter of a circle is prolonged through the circumference till the prolonged part is equal to one or other of the segments into which the perpendicular divides the diameter: what is the locus of the extremity of the perpendicular?

Ans. An ellipse.

14. The base and difference of the other two sides of a plane triangle are given: a circle touches one of these sides, as also the prolongation of the base and of the other side: what is the locus of the centre of the circle?

Ans. An hyperbola.

15. Required the locus of the centres of all the circles which pass through a fixed point, and touch a fixed straight line? *Ans.* A parabola.

It may be advisable, before closing this elementary treatise, to subjoin a few remarks having especial reference to the matters discussed in the foregoing section.

It has been satisfactorily shown that whenever an equation of the second degree, involving two variables, is presented to us as the equation of a curve, we can always pronounce with certainty, after a little examination of the equation, whether it represents an ellipse, an hyperbola, or a parabola: but, from the equation itself merely, we cannot deduce any *additional* information. The particular, or individual, ellipse, or hyperbola, or parabola, from which the equation has really been derived, could not be actually reproduced from that equation, unless, in addition to the equation itself, the axes of reference were also given: if these axes be unknown to us, if we are in ignorance as to whether they are rectangular or oblique, or, if though stated to be oblique, we are not informed of the degree of obliquity, we cannot enter into details as to the individual curve the equation of which has been submitted to us. A simple instance will suffice to make this clear, if indeed instances be wanted. Take the equation $y^2+x^2-mx=0$: then, without knowing anything about the axes of reference, we should pronounce the curve to be an ellipse; but we could not specify the *particular ellipse*, without knowing whether the axes of co-ordinates were rectangular or oblique. If they were stated to be rectangular, we could tell all about this particular ellipse: we should at once perceive that it was not in reality an ellipse proper, but a *circle*, the radius of which is $\frac{1}{2}$ *m*. But if the axes were declared to be oblique, we should know that, whatever its individual characteristics may be, it cannot be a *circle*: yet, unless we are informed what the degree of obliquity actually is, we cannot construct the individual ellipse; for the given equation refers it to axes of a certain definite degree of obliquity, and not to any other axes.

In the foregoing problems on *Loci*, nothing is left uncertain or ambiguous: the axes in each case are definitely fixed; and the equation of the locus is the equation which relates the several points of it to that individual pair of axes, and to no other pair.

The name *Conic Sections* is given to the assemblage of lines which have been the objects of consideration in the present volume, because each of them is a section of a cone, when cut by a plane; and because, whatever direction the cutting plane may take, the outline of the section can never be a curve other than one of those treated of under the appellation of *lines of the second order*. If an upright cone be cut by a plane perpendicular to its axis, that is, parallel to its base, the section will evidently be a *circle*: if, however, the plane be *inclined* to the axis, cutting through it obliquely, it is pretty obvious that the section will be an *ellipse*: but if the plane, instead of passing through the axis of the cone, be parallel to it, the section will be one of the branches of an *hyperbola*: the other branch would be the section of an opposite and equal cone, in an inverted position; each cone having a common apex, and the axes of the two being in one and the same straight line. If, lastly, the cutting plane pass through the axis in a direction parallel to the slant side of the cone, the section will be a *parabola*: these are the *only* curves which can be furnished by the various sections of a cone. And under this aspect it was that the ancient geometers investigated their properties.

But besides these curves of the second order, innumerable other curves may be traced out, as geometrical representatives of equations of the higher degrees: there are curves of the third order, the fourth order, the fifth order, &c.; according as the equations of them are of the third, fourth, or fifth degree, &c. As may be expected, the forms of some of them are complicated: many of them are figured, and the principal properties of them discussed, in the treatise of Maclaurin *On the general Properties of Geometrical Lines.* Several of these properties are very remarkable: we have room here to mention and prove only two.

1. Whatever two straight lines be taken for the axes of x and y, if two parallels to the axis of y be drawn through a curve of the nth order, and they so cut the axis of x, that the sum of the ordinates of the points where they cross the curve, on one side of the axis of x, be equal to the sum of the ordinates on the other side, for each of these parallels, then every other line parallel to the axis of y will cut the curve so that the sum of the ordinates on one side the axis of x shall likewise be equal to the sum of the ordinates on the other side.

Let the equation of the curve be written in the form

$$y^n+(ax+b)y^{n-1}+\ \dots\ =0\ \dots\ (1)$$

and let the equations of the two lines parallel to the axis of y be $x=p$, and $x=q$: then we must have

$$y^n+(ap+b)y^{n-1}+\ \dots\ =0,$$
$$\text{and } y^n+(aq+b)y^{n-1}+\ \dots\ =0.$$

Now if the sum of the negative values of y be equal to the sum of the positive values, the coefficients $(ap+b)$, and $(aq+b)$, must each be zero; since by the Theory of Equations, the coefficient of the second term, with changed sign, is always equal to the sum of the roots of the equation; and *here* this sum is zero: hence

$$ap+b=0,\text{ and } aq+b=0,\ \therefore\ a(p-q)=0,$$
$$\therefore\ a=0,\text{ and }\therefore\ b=0.$$

But if a and b, in (1), be each zero, the coefficient $(ax+b)$ must be zero, whatever be the value of x; and therefore, wherever a third parallel to the axis of y cuts the curve, the sum of the ordinates of the points of intersection with the curve must have the property announced.

2. If across a curve of any order (n), two secants, parallel to the sides of a fixed angle, be drawn from a point, the continued product of the parts intercepted between the curve and this point, of one of the secants, shall have the same constant ratio to the product of the parts of the other secant, however the situation of the point from which the two secants are drawn be varied.

Let the two secants be taken for the axes of reference: then for $y = 0$, and $x = 0$, the equation of the curve will give the conditions

$$ax^n + bx^{n-1} + cx^{n-2} + \ \ldots\ + px + q = 0 \ \ldots\ (1)$$
$$a'y^n + b'y^{n-1} + c'y^{n-2} + \ \ldots\ + p'y + q = 0 \ \ldots\ (2)$$

Now the roots of these equations will denote the parts of the two secants intercepted between the origin of the axes and the curve; and dividing the first equation by a, and the second by a', we shall have for the *product* of the roots of (1), the expression $\frac{q}{a}$; and for the product of the roots of (2), the expression $\frac{q}{a'}$; and since $\frac{q}{a} : \frac{q}{a'} :: a' : a$, the truth of the theorem may be inferred. For the removal of the origin, without disturbing the inclination of the axes, cannot cause any alteration in the values of a, a', the coefficients of the highest powers of x and y. The substitution of $x+h$ for x, and $y+h'$ for y, cannot affect the terms involving the *highest* power of x and of y, in any equation.

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